

Generalised Poisson-Dirichlet Distributions and the Negative Binomial Point Process

Yuguang Fan^{*1} and Ross A. Maller^{†2}

¹Australian Research Council Centre of Excellence for Mathematical and Statistical Frontiers, School of Mathematics and Statistics, University of Melbourne, Australia.

²Research School of Finance, Actuarial Studies & Statistics, the Australian National University, Australia.

December 8, 2016

Abstract

Take a Lévy process $(X_t)_{t \geq 0}$, order the positive jumps of X , and subtract the largest r of them at time $t > 0$ from X_t to get $^{(r)}X_t$ (r is a non-negative integer and $^{(0)}X_t = X_t$). When X is a subordinator, the sequence of ordered jumps, omitting the first r of them, and taken as proportions of their sum $^{(r)}X_t$, has a distribution on the infinite dimensional simplex, ∇_∞ . Assuming X is in the domain of attraction of a stable(α) law, this distribution has a limit which is the distribution of the corresponding stable proportions.

In the subordinator case these limiting proportions define a 2-parameter distribution on ∇_∞ which we call the $\text{PD}_\alpha^{(r)}$ distribution. When $r = 0$ it reduces to the PD_α distribution introduced by Kingman in 1975. We investigate some of its properties, in particular exploiting a connection with the negative binomial point process which allows us to analyse a size-biased version of $\text{PD}_\alpha^{(r)}$. As a consequence we can derive a stick-breaking representation for $\text{PD}_\alpha^{(r)}$. This program produces a large new class of distributions available for a variety of modelling purposes.

Keywords: generalised Poisson-Dirichlet laws; negative binomial point process; trimmed α -stable subordinator; trimmed Lévy processes; domain of attraction of stable laws.

2010 Mathematics Subject Classification: Primary 60G51, 60G52, 60G55; secondary 60G57.

^{*}Research supported by ARC Center of Excellence for Mathematical and Statistical Frontiers.

Corresponding author: yuguang.fan@unimelb.edu.au

[†]Research partially supported by ARC Grant DP1092502; Email: Ross.Maller@anu.edu.au

1 Introduction

Developments related to the Poisson-Dirichlet distribution and its generalisations have had an enormous impact in recent times, stimulating as well as synthesising a host of theoretical results connected in particular to the excursion theory of stochastic processes and to random partitions, and opening up a wealth of applications areas, especially for example in Bayesian statistics and population genetics. We refer to Bertoin (2006) and Feng (2010) for up-to-date accounts of various aspects.

To motivate the ideas that concern us here, start with a stable subordinator $(S_t)_{t \geq 0}$ of index $\alpha \in (0, 1)$ on \mathbb{R}^+ having jump process $(\Delta S_t := S_t - S_{t-})_{t > 0}$, and order the jumps as $\Delta S_t^{(1)} \geq \Delta S_t^{(2)} \geq \dots$. The random sequence $(\Delta S_1^{(i)}/S_1)_{i \geq 1}$ specifies a distribution on the infinite dimensional simplex ∇_∞ which we will refer to as a PD_α distribution. It was introduced by Kingman (1975) and subsequently gave rise to a large body of research. Of special interest to us are papers by Perman, Pitman and Yor (1992) and Pitman and Yor (1992; 1997) (hereafter, referred to as PPY (1992) and PY (1992; 1997)). They contain in particular formulae for the distribution of the *size-biased* vector¹ associated with PD_α .

The PD_α distribution arises by considering the ordered jumps $(\Delta S_t^{(i)})_{i \geq 1}$ and their relation to their sum, S_t . As a natural generalisation, delete the r largest jumps ($r \geq 1$ an integer) and consider the distribution of the remaining $(\Delta S_t^{(i)})_{i \geq r+1}$ taken as proportions of their sum, ${}^{(r)}S_t$, the latter being S_t with the r largest terms removed. Again we obtain a distribution on ∇_∞ , now with an extra parameter, r . When $r = 0$ (no trimming) this is a PD_α distribution, while for $r = 1, 2, \dots$, it defines a 2-parameter distribution on ∇_∞ which we call the $\text{PD}_\alpha^{(r)}$ distribution.

These results arise quite naturally by first considering a Lévy process (X_t) on \mathbb{R} in the domain of attraction of a stable distribution, then ordering the positive jumps of X . The ordered jumps taken as proportions of the original process then converge jointly to corresponding proportions based on the limiting stable random variables. We implement this procedure by applying formulae for the joint distributions of those proportions, which in turn are worked out from formulae previously given in Buchmann, Fan and Maller (2016) (hereafter, BFM (2016)) for distributions of trimmed Lévy processes.

The computation is done in quite a general setting; X need not be a subordinator, but when it is, the sequence of ordered proportions has a distribution on ∇_∞ , and when in addition X is in the domain of attraction of a stable(α) law, the limit is the distribution of the corresponding stable proportions, thus having a $\text{PD}_\alpha^{(r)}$ distribution.

¹PY (1997) used the notation $\text{PD}(\alpha, 0)$ where we use PD_α . They allow for an extra parameter in the distribution, different to our r . We discuss this point in Section 3.

Laplace transforms of the stable ratios take a reasonably explicit form, and in the subordinator case reveal a close connection with the *negative binomial point process* of Gregoire (1984). This suggests some rewarding new lines of enquiry, and we proceed to define a size-biased version of $\text{PD}_\alpha^{(r)}$ and use the point process representation to derive a corresponding stick-breaking representation.

This program produces a large new class of distributions represented as limits of underlying Lévy processes, available for a variety of modelling purposes.

1.1 Lévy Process Setup

Although for much of the paper we concentrate on subordinators, many of our results are quite general. So we consider an arbitrary real valued Lévy process $(X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with canonical triplet (γ, σ^2, Π) ; thus, having characteristic function $Ee^{i\theta X_t} = e^{t\Psi(\theta)}$, $t \geq 0$, $\theta \in \mathbb{R}$, with exponent

$$\Psi_X(\theta) := i\theta\gamma - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R} \setminus \{0\}} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}} \right) \Pi(dx). \quad (1.1)$$

Here $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a Lévy measure on \mathbb{R} , i.e., a Borel measure on \mathbb{R} with $\int_{\mathbb{R} \setminus \{0\}} (x^2 \wedge 1) \Pi(dx) < \infty$. The positive, negative and two-sided tails of Π are

$$\overline{\Pi}^+(x) := \Pi\{(x, \infty)\}, \quad \overline{\Pi}^-(x) := \Pi\{(-\infty, x)\}, \quad \text{and} \quad \overline{\Pi}(x) := \overline{\Pi}^+(x) + \overline{\Pi}^-(x), \quad x > 0. \quad (1.2)$$

Throughout, let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

Write $(\Delta X_t := X_t - X_{t-})_{t > 0}$, with $\Delta X_0 = 0$, for the jump process of X , and $\Delta X_t^{(1)} \geq \Delta X_t^{(2)} \geq \dots$ for the ordered jumps at time $t > 0$. Assume throughout that $\Pi\{\mathbb{R} \setminus \{0\}\} = \infty$, so there are infinitely many jumps, a.s., in any finite interval. Further assume $\overline{\Pi}^+(0+) > 0$, so the $\Delta X_t^{(i)}$ are positive a.s. for all $t > 0$ and $i \in \mathbb{N}$. Our objective in Section 2 is to study the (one-sided) “trimmed” process, by which we mean X_t minus its large (positive) jumps, at a given time t . Thus, the one-sided r -trimmed version of X is

$${}^{(r)}X_t := X_t - \sum_{i=1}^r \Delta X_t^{(i)}, \quad r \in \mathbb{N}, \quad t > 0 \quad (1.3)$$

(and we set ${}^{(0)}X_t \equiv X_t$). Detailed definitions and properties of this kind of ordering and trimming are given in Sections 4 and 5, where we identify the positive ΔX_t with the points of a Poisson point process on $[0, \infty)$.

In the next section our main task is to show that ratios formed by dividing ${}^{(r)}X_t$ by its ordered positive jumps converge to corresponding stable ratios when X is in the domain of attraction of a non-normal stable law.

2 Trimmed Lévy Processes and Ordered Lévy Jumps

Throughout this section the Lévy process X will be assumed to be in the domain of attraction of a non-normal stable random variable at 0 (or at ∞).² By this we mean that there are nonstochastic functions $a_t \in \mathbb{R}$ and $b_t > 0$ such that $(X_t - a_t)/b_t \xrightarrow{D} Y$, for an a.s. finite random variable Y , not degenerate at a constant, as $t \downarrow 0$. If this is the case then the limit random variable Y must be a stable random variable of index $\alpha \in (0, 2)$; equivalently, the Lévy tail $\bar{\Pi}(x)$ is regularly varying of index $-\alpha$ at 0, and the balance conditions

$$\lim_{x \downarrow 0} \frac{\bar{\Pi}^\pm(x)}{\bar{\Pi}(x)} = a_\pm, \quad (2.1)$$

where $a_+ + a_- = 1$, are satisfied. We consider one-sided (positive) trimming, so we always assume $a_+ > 0$.

Let $RV_0(\beta)$ ($RV_\infty(\beta)$) be the regularly varying functions of index $\beta \in \mathbb{R}$ at 0 (or ∞). Denote by $(S_t)_{t \geq 0}$ a stable process of index $\alpha \in (0, 2)$ having Lévy measure

$$\Lambda(dx) = \Lambda_S(dx) = -d(x^{-\alpha})\mathbf{1}_{\{x>0\}} + (a_-/a_+)d(-x)^{-\alpha}\mathbf{1}_{\{x<0\}}, \quad x \in \mathbb{R}, \quad (2.2)$$

with characteristic exponent³

$$\Psi_S(\theta) := \int_{\mathbb{R} \setminus \{0\}} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}} \right) \Lambda(dx). \quad (2.3)$$

Denote by $(\Delta S_t := S_t - S_{t-})_{t>0}$ the jump process of S and by

$$\Delta S_t^{(1)} \geq \Delta S_t^{(2)} \geq \dots \geq \Delta S_t^{(n)} \geq \dots$$

the ordered stable jumps at time $t > 0$. These are uniquely defined a.s. (no tied values a.s.) since the Lévy measure of S has no atoms. The positive and negative tails of Λ are $\bar{\Lambda}^+(x) = x^{-\alpha}$ and $\bar{\Lambda}^-(x) = (a_-/a_+)x^{-\alpha}$, for $x > 0$. Since $\bar{\Lambda}^+(0+) = \infty$, the $\Delta S_t^{(i)}$ are positive a.s., $i = 1, 2, \dots$

Define a centering function $\mu_X(\cdot)$ for X by

$$\mu_X(w) := \begin{cases} \gamma - \int_{[w,1]} x \Pi(dx), & 0 < w \leq 1, \\ \gamma + \int_{[-w,-1) \cup (1,w)} x \Pi(dx), & w > 1, \end{cases} \quad (2.4)$$

²The convergences in this section can be worked out as $t \downarrow 0$ or as $t \rightarrow \infty$. For definiteness and in keeping with modern trends in the area we supply the versions for $t \downarrow 0$, but little modification is needed for the case $t \rightarrow \infty$.

³It's convenient for us to write the stable characteristic exponent in this way. It can be put in a more usual form by adjusting the function a_t .

and similarly for $\mu_S(w)$ with γ taken as 0 and Λ replacing Π .

The main theorem in this section, Theorem 2.1, proves joint convergence of the trimmed process $(^{(r)}X_t)$, normalised by those large jumps trimmed, and after centering, to corresponding stable ratios. To state it, we need some further notation. Let $W = (W_v)_{v \geq 0}$ be a Lévy process on \mathbb{R} with triplet $(0, 0, \Lambda(dx)\mathbf{1}_{(-\infty, 1)})$, and let Γ_{r+n} denote a Gamma $(r+n)$ random variable independent of W . For each $n = 2, 3, \dots$ and $0 < u < 1$, suppose random variables $J_{n-1}^{(1)}(u) \geq J_{n-1}^{(2)}(u) \geq \dots \geq J_{n-1}^{(n-1)}(u)$ are distributed like the decreasing order statistics of $n-1$ independent and identically distributed random variables $(J_i(u))_{1 \leq i \leq n-1}$, each having the distribution

$$P(J_1(u) \in dx) = \frac{\Lambda(dx)\mathbf{1}_{\{1 \leq x \leq 1/u\}}}{1 - u^\alpha}, \quad x > 0; \quad (2.5)$$

and suppose $B_{r,n}$ is a Beta(r, n) random variable independent of the $(J_i(u))$. Define

$$\psi(\theta) = \int_{(-\infty, 1)} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}} \right) \Lambda(dx), \quad \theta \in \mathbb{R}, \quad (2.6)$$

and choose $\theta_0 > 0$ such that $|\psi(\theta)| < 1$ for $|\theta| \leq \theta_0$ (possible since $\psi(0) = 0$). Also define $\phi(\theta, u) = E e^{i\theta J_1(u)}$, $\theta \in \mathbb{R}$, with $J_1(u)$ having the distribution in (2.5); thus,

$$\phi(\theta, u) = (1 - u^\alpha)^{-1} \int_1^{1/u} e^{i\theta x} \Lambda(dx), \quad 0 < u < 1. \quad (2.7)$$

When $x_k > 0$, $1 \leq k \leq n-1$, $x_n = 1$, and $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$, write for shorthand

$$x_{n+} = \sum_{k=1}^n x_k \quad \text{and} \quad \tilde{\theta}_n = \tilde{\theta}_n(x_1, \dots, x_n) := \sum_{k=1}^n \frac{\theta_k}{x_k}, \quad (2.8)$$

and let $\int_{x \uparrow \geq 1}$ denote integration over the region $\{x_1 \geq x_2 \geq \dots \geq x_{n-1} \geq 1\} \subseteq \mathbb{R}^{n-1}$.

Theorem 2.1 which follows is a kind of multivariate version for a general Lévy of Theorem 1.1 of Kevei and Mason (2014) for a subordinator (see the remarks relating to our Eq. (2.16), below). They derive the limit distribution of the ratio of an r -trimmed subordinator (in our notation) to the $(r+1)$ -st largest jump, proving also a converse result, as well as boundary cases (slow and rapid variation). We apply Theorem 2.1 in Section 3, to get functional convergence of the generalised Poisson-Dirichlet distribution, as defined there, in the subordinator case.

Theorem 2.1. *Assume $\overline{\Pi} \in RV_0(-\alpha)$ for some $0 < \alpha < 2$ and (2.1).*

(i) *Then for each $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, as $t \downarrow 0$, we have the joint convergence*

$$\begin{aligned} & \left(\frac{^{(r)}X_t - t\mu_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+1)}}, \dots, \frac{^{(r)}X_t - t\mu_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+n)}} \right) \\ & \xrightarrow{D} \left(\frac{^{(r)}S_1 - \mu_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+1)}}, \dots, \frac{^{(r)}S_1 - \mu_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+n)}} \right). \end{aligned} \quad (2.9)$$

(ii) When $r \in \mathbb{N}_0$, $n = 2, 3, \dots$, the random vector on the RHS of (2.9) has characteristic function which can be represented, for $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$, as

$$\begin{aligned} & \mathbb{E} \exp \left(i \sum_{k=1}^n \frac{\theta_k ({}^{(r)}S_1 - \mu_S(\Delta S_1^{(r+n)}))}{\Delta S_1^{(r+k)}} \right) = \\ & \int_{\mathbf{x}^\dagger \geq 1} e^{i\tilde{\theta}_n x_n +} \mathbb{E}(e^{i\tilde{\theta}_n W_{\Gamma_{r+n}}}) \mathbb{P}(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1); \end{aligned} \quad (2.10)$$

or, alternatively, when $\max_{1 \leq k \leq n} |\theta_k| \leq \theta_0$, it can be written as

$$\int_{\mathbf{x}^\dagger \geq 1} \frac{e^{i\tilde{\theta}_n x_n +}}{(1 - \psi(\tilde{\theta}_n))^{r+n}} \mathbb{P}(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1). \quad (2.11)$$

(iii) When $r \in \mathbb{N}_0$, $n \in \mathbb{N}$ we have

$$\frac{{}^{(r)}X_t - t\mu_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+n)}} \xrightarrow{\mathbb{D}} \frac{{}^{(r)}S_1 - \mu_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+n)}}, \text{ as } t \downarrow 0, \quad (2.12)$$

where the random variable on the RHS of (2.12) has characteristic function

$$\frac{e^{i\theta}}{(1 - \psi(\theta))^{r+n}} \mathbb{E}(\phi^{n-1}(\theta, B_{r,n}^{1/\alpha})), \quad \theta \in \mathbb{R}. \quad (2.13)$$

Setting $n = 1$ in (2.12), and (since ${}^{(r)}X_t / \Delta X_t^{(r+1)} = 1 + {}^{(r+1)}X_t / \Delta X_t^{(r+1)}$) replacing $r + 1$ by r gives

Corollary 2.1. For each $r \in \mathbb{N}$, $\theta \in \mathbb{R}$, $|\theta| \leq \theta_0$,

$$\frac{{}^{(r)}X_t - t\mu_X(\Delta X_t^{(r)})}{\Delta X_t^{(r)}} \xrightarrow{\mathbb{D}} \frac{{}^{(r)}S_1 - \mu_S(\Delta S_1^{(r)})}{\Delta S_1^{(r)}}, \text{ as } t \downarrow 0, \quad (2.14)$$

where

$$\mathbb{E}(e^{i\theta({}^{(r)}S_1 - \mu_S(\Delta S_1^{(r)}))/\Delta S_1^{(r)}}) = \mathbb{E}(e^{i\theta W_{\Gamma_r}}) = \frac{1}{(1 - \psi(\theta))^r}. \quad (2.15)$$

Further, $({}^{(r)}S_1 - \mu_S(\Delta S_1^{(r)}))/\Delta S_1^{(r)} \stackrel{\mathbb{D}}{=} W_{\Gamma_r}$, being a Gamma-subordinated Lévy process, is infinitely divisible for each $r \in \mathbb{N}$.

The unwieldy centering functions μ_X and μ_S in (2.9)–(2.15) can be simplified in many cases. Most relevant for us, when X is a driftless subordinator, μ_X can be replaced by the drift d_X of X , and without loss of generality we can assume $d_X = 0$. The convergences in (2.9)–(2.14) can then be written in terms of Laplace transforms. Then we get the result in Theorem 1.1 of Kevei and Mason (2014): *assume X is a driftless subordinator in the domain of attraction (at 0) of a stable random variable with index $\alpha \in (0, 1)$. Then for $r \in \mathbb{N}$*

$$\frac{{}^{(r)}X_t}{\Delta X_t^{(r)}} \xrightarrow{\mathbb{D}} {}^{(r)}Y, \text{ as } t \downarrow 0, \quad (2.16)$$

where $(r)Y$ is a finite non-degenerate random variable. From Theorem 2.1 we can identify $(r)Y$ as having the distribution of $(r)S_1/\Delta S_1^{(r)}$. Kevei and Mason show, conversely, in this subordinator case, that when (2.16) holds with $(r)Y$ a finite non-degenerate random variable, then X is in the domain of attraction (at 0) of a stable random variable with index $\alpha \in (0, 1)$. They also give a formula for the Laplace transform of $(r)Y$. We can state this as: *suppose (2.16) holds. Then (2.15) becomes*

$$\mathbb{E}(e^{-\lambda(r)S_1/\Delta S_1^{(r)}}) = \mathbb{E}(e^{-\lambda W_{\Gamma_r}}) = \frac{1}{(1 + \Psi(\lambda))^r}, \quad r \in \mathbb{N}, \quad (2.17)$$

where now $W = (W_v)_{v \geq 0}$ is a driftless subordinator with measure $\Lambda(dx)\mathbf{1}_{(0,1)}$, and

$$\Psi(\lambda) = \int_{(0,1)} (1 - e^{-\lambda x}) \Lambda(dx), \quad \lambda > 0. \quad (2.18)$$

The form of the Laplace transform in (2.17) suggests a connection with the negative binomial point process which we develop as a main theme in Subsection 3.2.

To lead into Section 3, we continue to consider the case when X is a driftless subordinator. Our final result in this section shows that ratios of the form $(r+n)X_t/\Delta X_t^{(r)}$ have remarkable stability properties. In the next theorem the remainder after removing an increasing number of jumps, $r+n$, from X is shown to be small order $\Delta X_t^{(r)}$, a.s., as $n \rightarrow \infty$, uniformly on compacts.

Theorem 2.2. *Suppose X is a driftless subordinator with $\overline{\Pi} \equiv \overline{\Pi}^+ \in RV_0(-\alpha)$ for some $0 < \alpha < 1$. Then for each $r \in \mathbb{N}$*

$$\frac{(r+n)X_t}{\Delta X_t^{(r)}} \rightarrow 0, \quad \text{a.s., as } n \rightarrow \infty, \quad (2.19)$$

uniformly in $t \in (0, t_0]$, for any $t_0 > 0$.

Remark 2.1. The convergence $(X_t - a_t)/b_t \xrightarrow{D} Y$ to a stable rv further implies process convergence in the Skorokhod J_1 -topology. This suggests a functional version of Theorem 2.1. We defer exploration of this and other point process convergence questions arising from the results in the next sections to another time.

3 Normalised Stable Subordinator Jumps

Continue to assume that X_t is a subordinator with no drift such that $\overline{\Pi} \in RV_0(-\alpha)$ with $\alpha \in (0, 1)$, and, now, that $(S_t)_{t>0}$ is the driftless stable subordinator with ordered jumps $(\Delta S_t^{(i)})_{t>0}$, $i \in \mathbb{N}$, having Lévy measure

$$\Lambda(dx) = c\alpha x^{-\alpha-1}dx, \quad \text{for some } c > 0 \text{ and } 0 < \alpha < 1 \quad (3.1)$$

(a variant of the Λ in (2.2) but we use the same notation). In the next subsection we define the $\text{PD}_\alpha^{(r)}$ distribution as that of the sequence of ratios $(\Delta S_1^{(i)}/(r)S_1)_{i \geq r+1}$.

3.1 Generalised Poisson-Dirichlet Distributions

Fix $r \in \mathbb{N}_0$ and define

$$V_n^{(r)}(t) := \frac{\Delta X_t^{(r+n)}}{(r)X_t}, \quad n \in \mathbb{N}, \quad t > 0,$$

with $V_n(t) := V_n^{(0)}(t) = \Delta X_t^{(n)}/X_t$. Consider the sequence

$$(V_n^{(r)}(t))_{n \in \mathbb{N}}$$

as a stochastic process in discrete time, $n \in \mathbb{N}$, indexed by $t > 0$. By Theorem 2.1 we have the finite dimensional convergence

$$(V_{n_1}^{(r)}(t), V_{n_2}^{(r)}(t), \dots, V_{n_k}^{(r)}(t)) \xrightarrow{D} (V_{n_1}^{(r)}, V_{n_2}^{(r)}, \dots, V_{n_k}^{(r)}),$$

as $t \downarrow 0$, for any integers $r \in \mathbb{N}$ and $(n_1, n_2, \dots, n_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$, where

$$V_n^{(r)} := \frac{\Delta S_1^{(r+n)}}{(r)S_1}, \quad n \in \mathbb{N}.$$

The finite dimensional convergence suffices for weak convergence of $(V_n^{(r)}(t))$ to $(V_n^{(r)})$ as $t \downarrow 0$ (see e.g. Daley and Vere-Jones (1988, Thm. 9.1.VI, p.274)).

Further, for each $r \in \mathbb{N}$ the series $\sum_{n \geq 1} V_n^{(r)}(t)$ converges a.s., uniformly in $t \in (0, t_0]$, for any $t_0 > 0$. This derives from Theorem 2.2 as follows. For $n, m \in \mathbb{N}$, $m > n$, we have

$$\sum_{n < j \leq m} V_j^{(r)}(t) \leq \frac{(r+n)X_t}{(r)X_t} \leq \frac{(r+n)X_t}{\Delta X_t^{(r+1)}}, \quad (3.2)$$

and by (2.19) the RHS tends to 0 a.s. as $n \rightarrow \infty$, uniformly in $t \in (0, t_0]$.⁴ For the stable version,

$$\sum_{n < j \leq m} V_j^{(r)} \leq \frac{(r+n)S_1}{(r)S_1} \leq \frac{(r+n)S_1}{\Delta S_1^{(r+1)}} \rightarrow 0, \quad \text{a.s., as } n \rightarrow \infty.$$

Clearly all of $\sum_{n \geq 1} V_n^{(r)}$ and $\sum_{n \geq 1} V_n^{(r)}(t)$, for each $t > 0$, have value 1.

The distributions of $(V_n^{(r)}(t))_{n \in \mathbb{N}}$ (for a given $t > 0$) and $(V_n^{(r)})_{n \in \mathbb{N}}$ when $r \in \mathbb{N}$ define new families of distributions on ∇_∞ derived from subordinators X and S . The case of the stable limit, especially, is amenable to further analysis along the lines previously worked out for PD_α . This is developed further in the next section.

⁴When X is a subordinator the series $\sum_{0 < s \leq t} \Delta X_s$ converges a.s. uniformly in $t \in (0, t_0]$ for any $t_0 > 0$ as is shown in Sato (1999, Thm.19.2, p.120). But we assert the a.s. uniform convergence of the series of ratios in (3.2). The uniformity is only for bounded t ; we would not expect uniform convergence in intervals $[t_0, \infty)$.

Definition 3.1. Let $(S_t, 0 \leq t \leq 1)$ be a driftless stable subordinator with index $\alpha \in (0, 1)$ and take $r \in \mathbb{N}_0$. Then the distribution of the sequence

$$(V_n^{(r)})_{n \in \mathbb{N}} = (V_1^{(r)}, V_2^{(r)}, \dots) = \left(\frac{\Delta S_1^{(r+1)}}{{}^{(r)}S_1}, \frac{\Delta S_1^{(r+2)}}{{}^{(r)}S_1}, \dots \right) \quad (3.3)$$

we call a $\text{PD}_\alpha^{(r)}$ distribution. When $r = 0$, PD_α is recovered.

Remark 3.1. Our approach is by the “trimming” of large jumps from the stable subordinator. By contrast, processes of “deletion” and “renormalisation” of the first r size-biased picks derived from the jumps of the (untrimmed) subordinator play prominent roles in Pitman (2003) and PY (1997, Prop. 34, 35). The $\text{PD}_\alpha^{(r)}$ of Definition 3.1 is distinct from these kinds of laws. $\text{PD}_\alpha^{(r)}$ is obtained from the deletion of the r largest jumps of S_1 , followed by renormalisation, rather than from the deletion of the first r size-biased picks from $(\Delta S_1^{(i)})$. This results in a different dependence structure in the stick-breaking representation (cf. Theorem 3.1 in the next section).

Results related to those of PY (1997) concerning deletion of excursion intervals of certain Bessel bridges are in PPY (1992, Sect.3); see also James (2013; 2015).

Remark 3.2. Similar to (3.3), any distribution on ∇_∞ with a subordinator representation can be generalised by removing the r largest jumps up till time $t > 0$ from the subordinator. For example:

- (i) the usual Poisson-Dirichlet distribution, denoted as $\text{PD}(0, \theta)$ in PY (1997), can be generalised by trimming a Gamma subordinator up till time $\theta > 0$;
- (ii) the two parameter generalised Poisson-Dirichlet distribution $\text{PD}(\alpha, \theta)$ in PY (1997) can be extended by trimming a generalised Gamma subordinator up till a random time mixed with a $\Gamma_{\theta/\alpha}$ distribution (see PY (1997, Prop. 21)).

We do not pursue these generalisations here, exploring instead a connection with the negative binomial process in the next subsection, but we conclude this subsection with formulae for the Laplace transforms of the $V_n^{(r)}$. Recall $\phi(\theta, u)$ from (2.7) and $\Psi(\lambda)$ from (2.18). The following formulae follow immediately from (2.11) and (2.13).

Proposition 3.1. Fix $r, n \in \mathbb{N}$ and $\lambda_i > 0$, $1 \leq i \leq n$, and suppose $(V_n^{(r)})$ follows a $\text{PD}_\alpha^{(r)}$ distribution. Then, with the notation from Theorem 2.1 and Eq. (2.18),

$$\begin{aligned} & \mathbb{E}(e^{-\sum_{i=1}^n \lambda_i / V_i^{(r)}}) \\ &= \int_{\mathbf{x}^\dagger \geq 1} \frac{e^{-\tilde{\lambda}_n x_n}}{(1 + \Psi(\tilde{\lambda}_n))^{r+n}} \mathbb{P}(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1), \end{aligned}$$

where $\tilde{\lambda}_n = \tilde{\lambda}_n(x_1, \dots, x_n) := \sum_{k=1}^n \lambda_k / x_k$; and in particular

$$\mathbb{E}(e^{-\lambda / V_n^{(r)}}) = \frac{e^{-\lambda} \mathbb{E}[\phi(i\lambda, B_{r,n}^{1/\alpha})^{n-1}]}{(1 + \Psi(\lambda))^{r+n}}, \quad \lambda > 0. \quad (3.4)$$

Remark 3.3. When $r = 0$ we can formally set $B_{r,n} = 0$ and $\phi(i\lambda, 0) = \int_1^\infty e^{-\lambda x} \Lambda(dx)$ in (3.4), and then apart from a normalising constant the formula agrees with PY (1997, Eq. (38)) for the PD_α distribution.

We write the density function of a Gamma random variable with parameter r as

$$\mathbb{P}(\Gamma_r \in dx) = \frac{x^{r-1} e^{-x} dx}{\Gamma(r)} \mathbf{1}_{\{x>0\}}.$$

The Beta random variable $B_{a,b}$ with parameters $a, b > 0$ has density

$$f_B(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{\{0<x<1\}} = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{\{0<x<1\}}, \quad (3.5)$$

where $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$ is the Gamma function and $B(a, b)$ is the Beta function.

3.2 The Negative Binomial Process and the Distribution of $\text{PD}_\alpha^{(r)}$

In this subsection we connect the results of Theorem 2.1 with the negative binomial point process introduced by Gregoire (1984). As previously, S is the driftless stable subordinator with index $\alpha \in (0, 1)$. It turns out that to generate the Laplace transform in (2.17), we have to construct a point process from ratios of stable jumps, rather than from the jumps themselves. Thus, for $r \in \mathbb{N}$, and with δ_x denoting a point mass at $x \in \mathbb{R}$, define a random point measure on the Borel sets of $(0, 1)$ by

$$\mathbb{B}^{(r)} = \sum_{i \geq 1} \delta_{J_r(i)}, \quad \text{where} \quad J_r(i) = \frac{\Delta S_1^{(r+i)}}{\Delta S_1^{(r)}}, \quad i = 1, 2, \dots \quad (3.6)$$

Let $(\mathbb{M}, \mathcal{M})$ be the space of all point measures⁵ on $(0, 1)$ with its usual Borel σ -algebra and let \mathcal{F}^+ be the set of nonnegative measurable functions on $(0, 1)$. A random measure ξ on $(\mathbb{M}, \mathcal{M})$ has Laplace functional defined as

$$\Phi(f) = \mathbb{E}(e^{-\xi(f)}) = \int_{M \in \mathbb{M}} e^{-\int_{(0,1)} f(x) M(dx)} \mathbb{P}(\xi \in dM), \quad f \in \mathcal{F}^+.$$

Given a measure Π on $(0, \infty)$, locally finite at infinity, Gregoire (1984) defines the point process $\mathcal{BN}(r, \Pi)$ on $(\mathbb{M}, \mathcal{M})$ in terms of its Laplace functional as

$$\Phi(f) = \left(1 + \int_{(0,\infty)} (1 - e^{-f(x)}) \Pi(dx) \right)^{-r}, \quad f \in \mathcal{F}^+. \quad (3.7)$$

Recall (3.1) and let $\tilde{\Lambda}(dx) := \alpha x^{-\alpha-1} dx \mathbf{1}_{\{0<x<1\}}$ be the normalised $\Lambda(dx)$ restricted to $(0, 1)$. For each $r \in \mathbb{N}$, denote the law of $\mathcal{BN}(r, \tilde{\Lambda})$ by \mathbb{P}_r , that is

$$\mathbb{P}_r(dM) = \mathbb{P}(\mathcal{BN}(r, \tilde{\Lambda}) \in dM), \quad M \in \mathbb{M}.$$

⁵We generally follow the exposition in Resnick (1987, Chap. 3) for the following setup.

Denote the family of Palm distributions of P_r by $P_r^{(x)}$, $x \in (0, 1)$.

The correspondence between (3.7) and the righthand side of (2.17) suggests the following proposition.

Proposition 3.2. *Let $\mathbb{B}^{(r)}$ be defined as in (3.6). Then*

- (i) $\mathbb{B}^{(r)}$ is a negative binomial point process with distribution P_r such that
- (ii) $E(\mathbb{B}^{(r)}(A)) = r\tilde{\Lambda}(A)$ for any Borel set $A \subset (0, 1)$.
- (iii) The Laplace functional of the probability measure $P_r^{(x)}$ on $(\mathbb{M}, \mathcal{M})$ satisfies

$$\Phi_{P_r^{(x)}}(f) = \Phi_{\delta_x}(f)\Phi_{P_{r+1}}(f), \quad f \in \mathcal{F}^+. \quad (3.8)$$

Remark 3.4 (Interpretation of $P_r^{(x)}$). The Palm distribution $P_r^{(x)}$ is the conditional distribution of $\mathcal{BN}(r, \tilde{\Lambda})$ given $\mathcal{BN}(r, \tilde{\Lambda})(\{x\}) > 0$. From (3.8), we can interpret $P_r^{(x)}$ in the following way. Let ξ be distributed as $\mathcal{BN}(r+1, \tilde{\Lambda})$. Then $P_r^{(x)}$ is the distribution of $\xi + \delta_x$.

Proof of Proposition 3.2: (i): Conditional on $\{\Delta S_1^{(r)} = v\}$, $v > 0$, the truncated point process $\{\Delta S_1^{(r+j)}, j \in \mathbb{N}\}$ is a Poisson point process with intensity measure $\Lambda(dx)1\{x < v\}$. Anticipating (4.1), we can write the Laplace functional of $\mathbb{B}^{(r)}$ as

$$\begin{aligned} E(e^{-\mathbb{B}^{(r)}(f)}) &= \int_{v>0} \exp\left(-\int_0^v (1 - e^{-f(x/v)})\Lambda(dx)\right) P(\overline{\Lambda}^{\leftarrow}(\Gamma_r) \in dv) \\ &= \int_{v>0} \exp\left(-\int_0^1 (1 - e^{-f(x)})\Lambda(vdx)\right) P(\overline{\Lambda}^{\leftarrow}(\Gamma_r) \in dv) \\ &= \int_{v>0} \exp\left(-cv^{-\alpha} \int_0^1 (1 - e^{-f(x)})\alpha x^{-1-\alpha} dx\right) P(\Gamma_r \in d(cv^{-\alpha})), \end{aligned} \quad (3.9)$$

for each $f \in \mathcal{F}^+$. By change of variable in (3.9) with $y = cv^{-\alpha}$, we have

$$\begin{aligned} E(e^{-\mathbb{B}^{(r)}(f)}) &= \int_0^\infty \exp\left(-y \int_0^1 (1 - e^{-f(x)})\tilde{\Lambda}(dx)\right) P(\Gamma_r \in dy) \\ &= \int_0^\infty \exp\left(-y \int_0^1 (1 - e^{-f(x)})\tilde{\Lambda}(dx)\right) \frac{y^{r-1}e^{-y}}{\Gamma(r)} dy \\ &= \left(1 + \int_0^1 (1 - e^{-f(x)})\tilde{\Lambda}(dx)\right)^{-r}. \end{aligned} \quad (3.10)$$

Comparing (3.10) with (3.7) proves Part (i). Parts (ii) and (iii) follow from Part (i) by Props. 3.3 and 4.3 in Gregoire (1984). \square

Remark 3.5. (i) The sum of the points in $\mathbb{B}^{(r)}$ is ${}^{(r)}S_1/\Delta S_1^{(r)}$, hence the connection with (2.17).

(ii) A variety of formulae relating to the Poisson-Dirichlet distributions have been derived over the years, including an iterative formula for the joint density of the first n terms of PD_α (Perman (1993, Thm.2)). Such formulae, while explicit, are “rather intractable” (PY (1992, p.329)), and simpler structures can be revealed for the corresponding size-biased permutation; see, e.g., PPY (1992, Thm 1.2) (attributed to Perman (1990)), which allows for a “stick-breaking” representation of PD_α in terms of independent Beta rvs. This motivates us to consider the size-biased permutation of $\text{PD}_\alpha^{(r)}$ and to investigate a stick-breaking-like representation in the r -trimmed case through the random point measure $\mathbb{B}^{(r)}$.

Pitman (1995) proved that the $\text{PD}(\alpha, \theta)$ of PY (1997) is the largest class of distributions with a stick-breaking representation in terms of independent beta rvs; inevitably, then, our enlarged class $\text{PD}_\alpha^{(r)}$ requires a dependent stick-breaking representation. (James (2013) derives another class of distributions, $\text{PG}(\alpha, \zeta)$, from mixing generalised Gamma subordinators, which also has a dependent stick-breaking representation.) The dependence structure will become clear in the main Theorem 3.1 of this section which gives a formula for the density of the size-biased version of the sequence $(V_n^{(r)})$ in (3.3). The remaining calculations in this section lead up to Theorem 3.1.

Henceforth fix $r \in \mathbb{N}$. Write

$$\mathfrak{J}_r := \{J_r(1), J_r(2), J_r(3), \dots\} \quad (3.11)$$

for the points of $\mathbb{B}^{(r)}$, with sum

$${}^{(r)}T := \sum_{i \geq 1} J_r(i) = \frac{{}^{(r)}S_1}{\Delta S_1^{(r)}}. \quad (3.12)$$

Define the *size-biased random permutation* of \mathfrak{J}_r , denoted by $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \dots)$, in the following way. Conditional on \mathfrak{J}_r , the first term \tilde{J}_1 takes values among the members of \mathfrak{J}_r with probabilities

$$\mathbb{P}(\tilde{J}_1 = J_r(i) \mid \mathfrak{J}_r) = \frac{J_r(i)}{\sum_{\ell \geq 1} J_r(\ell)} = \frac{\Delta S_1^{(r+i)}}{{}^{(r)}S_1}, \quad i = 1, 2, \dots$$

Conditional on \mathfrak{J}_r and $\tilde{J}_1, \dots, \tilde{J}_{n-1}$, for each $n = 2, 3, \dots$, the n th term \tilde{J}_n takes values among $\{J_r(j), j = 1, 2, \dots; J_r(j) \neq \tilde{J}_l, l = 1, \dots, n-1\}$, with probabilities

$$\mathbb{P}(\tilde{J}_n = J_r(j) \mid \tilde{J}_1, \dots, \tilde{J}_{n-1}, \mathfrak{J}_r) = \frac{\Delta S_1^{(r+j)} \mathbf{1}\{\Delta S_1^{(r+j)} \neq \tilde{J}_l \cdot \Delta S_1^{(r)}, l = 1, \dots, n-1\}}{{}^{(r)}S_r - \Delta S_1^{(r)} \cdot (\sum_{l=1}^{n-1} \tilde{J}_l)}.$$

Then the sums of the remaining points in the point process, after removing points by size-biased sampling, are

$${}^{(r)}T_1 := {}^{(r)}T - \tilde{J}_1, \quad \text{and for each } n > 1, \quad {}^{(r)}T_n := {}^{(r)}T_{n-1} - \tilde{J}_n. \quad (3.13)$$

The successive *residual fractions* are

$${}^{(r)}U_1 := \frac{{}^{(r)}T_1}{{}^{(r)}T} = 1 - \frac{\tilde{J}_1}{{}^{(r)}T}, \quad (3.14)$$

and for each $n > 1$,

$${}^{(r)}U_n := \frac{{}^{(r)}T_n}{{}^{(r)}T_{n-1}} = 1 - \frac{\tilde{J}_n}{{}^{(r)}T_{n-1}}. \quad (3.15)$$

For M a point measure in \mathbb{M} , let $T(M) = \sum_{x \in M} x$ be the sum of the magnitudes of the points in M . For each $i \in \mathbb{N}_0$ let the density of $T(\mathbb{B}^{(r+i)})$ with $\mathbb{B}^{(r+i)}$ distributed as $\mathcal{BN}(r+i, \tilde{\Lambda})$ be

$$g_{r+i}(t) := P(T(\mathbb{B}^{(r+i)}) \in dt) / dt = P({}^{(r+i)}T \in dt) / dt. \quad (3.16)$$

By (3.12), ${}^{(r)}T = {}^{(r)}S_1 / \Delta S_1^{(r)}$, so by (2.17), g_{r+i} satisfies, for $\lambda > 0$, $r \in \mathbb{N}$, $i \in \mathbb{N}_0$,

$$\int_0^\infty e^{-\lambda x} g_{r+i}(x) dx = \left(1 + \int_0^1 (1 - e^{-\lambda x}) \tilde{\Lambda}(dx)\right)^{-r-i}. \quad (3.17)$$

Alternatively expressed, g_{r+i} is the density of $\widetilde{W}_{\Gamma_{r+i}}$, where $(\widetilde{W}_v)_{v \geq 0}$ is the driftless subordinator with Lévy measure $\tilde{\Lambda}(dx)$.

The next lemma derives important properties of $\mathbb{B}^{(r)}$. It will be apparent that our proofs owe much to the methods of PPY (1992), PY (1992) and Fitzsimmons, Pitman and Yor (1993, Sect. 5). In a remark at the end of this section we discuss briefly the differences as well as the similarities between our approaches.

Proposition 3.3. *For $r \in \mathbb{N}$ let $\mathbb{B}^{(r)}$ be a negative binomial point process with distribution P_r . Let $\mathbb{B}_1^{(r)} = \mathbb{B}^{(r)} - \tilde{J}_1$ be the remaining process after removing the first size-biased pick.*

(i) *Then for $0 < x < 1$, $M \in \mathbb{M}$,*

$$P(\tilde{J}_1 \in dx, \mathbb{B}_1^{(r)} \in dM) = \frac{x}{T(M) + x} r \tilde{\Lambda}(dx) P(\mathbb{B}^{(r+1)} \in dM). \quad (3.18)$$

(ii) *For $0 < x < 1$, $M \in \mathbb{M}$, $t > 0$, we have*

$$P(\tilde{J}_1 \in dx, \mathbb{B}_1^{(r)} \in dM, {}^{(r)}T_1 \in dt) = \frac{x}{t+x} r \tilde{\Lambda}(dx) P(\mathbb{B}^{(r+1)} \in dM, T(\mathbb{B}^{(r+1)}) \in dt). \quad (3.19)$$

(iii) *For $0 < x < 1$, $t > 0$, we have*

$$P(\tilde{J}_1 \in dx, {}^{(r)}T_1 \in dt) = \frac{x}{t+x} r \tilde{\Lambda}(dx) P({}^{(r+1)}T \in dt). \quad (3.20)$$

Remark 3.6. Compare the results of Proposition 3.3 with the corresponding Poisson case in Lemma 2.2 of PPY (1992).

Proof of Proposition 3.3: (i) By the definition of the size-biased picks, we have

$$P(\tilde{J}_1 \in dx \mid \mathbb{B}^{(r)} = M) = \frac{x}{T(M)} M(dx), \quad 0 < x < 1, \quad M \in \mathbb{M} \setminus \{\phi\}. \quad (3.21)$$

We use the following property of *Palm distributions* (see for instance Daley and Vere-Jones (1988, Sect. 12.1)):

$$r\tilde{\Lambda}(dx)P_r^{(x)}(dM) = M(dx)P_r(dM) = M(dx)P(\mathbb{B}^{(r)} \in dM) \quad (3.22)$$

(noting that the first moment measure of $\mathbb{B}^{(r)}$ is $r\tilde{\Lambda}(dx)$, by Prop. 3.2 (ii)). Write $P_{r+i}(dM) = P(\mathbb{B}^{(r+i)} \in dM)$ for $i \in \mathbb{N}_0$ and $M \in \mathbb{M}$. Then, from (3.21) and (3.22),

$$P(\tilde{J}_1 \in dx, \mathbb{B}^{(r)} \in dM) = \frac{x}{T(M)} M(dx)P_r(dM) = \frac{x}{T(M)} r\tilde{\Lambda}(dx)P_r^{(x)}(dM). \quad (3.23)$$

By (3.8), $P_r^{(x)}$ is the distribution of $\delta_x + \xi$ where ξ is distributed as $\mathcal{BN}(r+1, \tilde{\Lambda})$. For each $x \in (0, 1)$, let $\mathbb{B}^{(r, x-)} = \mathbb{B}^{(r)} - \delta_x$. Changing variable to $M_1 = M - \delta_x$ in (3.23) gives (see e.g. PY (1992, Lemma 2.2))

$$P(\tilde{J}_1 \in dx, \mathbb{B}^{(r, x-)} \in dM_1) = \frac{x}{T(M_1) + x} r\tilde{\Lambda}(dx)P_{r+1}(dM_1). \quad (3.24)$$

Then noting that, jointly, $P(\tilde{J}_1 \in dx, \mathbb{B}_1^{(r)} \in dM_1) = P(\tilde{J}_1 \in dx, \mathbb{B}^{(r, x-)} \in dM_1)$, we have proved (3.18).

(ii) ${}^{(r)}T_1 = T(\mathbb{B}_1^{(r)})$ is a deterministic transformation of $\mathbb{B}_1^{(r)}$, so

$$\begin{aligned} P(\tilde{J}_1 \in dx, \mathbb{B}_1^{(r)} \in dM, T(\mathbb{B}_1^{(r)}) \leq y) &= P(\tilde{J}_1 \in dx, \mathbb{B}_1^{(r)} \in dM, M \in Q_y) \\ &= \mathbf{1}_{\{M \in Q_y\}} P(\tilde{J}_1 \in dx, \mathbb{B}_1^{(r)} \in dM), \end{aligned}$$

where $Q_y := \{M \in \mathbb{M} : T(M) \leq y\}$, $y > 0$. By (3.18) the last expression equals

$$\frac{x}{T(M) + x} r\tilde{\Lambda}(dx)P(\mathbb{B}^{(r+1)} \in dM, T(\mathbb{B}^{(r+1)}) \leq y),$$

from which (3.19) follows.

(iii) Integrating M out of (3.19) and recalling (3.16) gives (3.20) via

$$P(\tilde{J}_1 \in dx, T(\mathbb{B}_1^{(r)}) \in dt) = \frac{x}{t+x} r\tilde{\Lambda}(dx)P(T(\mathbb{B}^{(r+1)}) \in dt).$$

This completes the proof of Proposition 3.3. \square

We can now compute the joint density of the size-biased points of $\mathcal{BN}(r, \tilde{\Lambda})$. Write the ascending factorial of base r and order n as $r^{(n)} = r(r+1) \cdots (r+n-1)$, $n \in \mathbb{N}$.

Proposition 3.4. Fix $r, n \in \mathbb{N}$. Given $x_i \in (0, 1)$, $1 \leq i \leq n$, $x_i \neq x_j$ for $i \neq j$, and $t > \sum_{i=1}^n x_i$, we have (interpreting $\sum_0^1 \equiv 0$)

$$\begin{aligned} & \mathbb{P}(\tilde{J}_1 \in dx_1, \dots, \tilde{J}_n \in dx_n, {}^{(r)}T \in dt) \\ &= r^{(n)} \alpha^n \prod_{i=1}^n \frac{x_i^{-\alpha} dx_i}{t - \sum_{j=1}^{i-1} x_j} \mathbb{P}\left({}^{(r+n)}T \in d\left(t - \sum_{i=1}^n x_i\right)\right). \end{aligned} \quad (3.25)$$

Proof of Proposition 3.4: Given $x_1, \dots, x_n \in (0, 1)$, $x_i \neq x_j$ for $i \neq j$, and $M \in \mathbb{M}$, write $M_{i+1} = M_i - \delta_{x_{i+1}}$, with $M_0 = M$ and $i = 0, \dots, n-1$.

We consider only the first two size-biased picks with $x_1 \neq x_2$. The extension to general n is similar. Note that

$$\begin{aligned} & \mathbb{P}\{\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, \mathbb{B}_1^{(r)} \in dM_1, T(\mathbb{B}_1^{(r)}) \in d(t - x_1)\} \\ &= \mathbb{P}(\tilde{J}_2 \in dx_2 \mid \tilde{J}_1 = x_1, \mathbb{B}_1^{(r)} = M_1, {}^{(r)}T_1 = t - x_1) \\ & \quad \times \mathbb{P}(\tilde{J}_1 \in dx_1, \mathbb{B}_1^{(r)} \in dM_1, {}^{(r)}T_1 \in d(t - x_1)) \\ &= \mathbb{P}(\tilde{J}_2 \in dx_2 \mid \mathbb{B}_1^{(r)} = M_1, {}^{(r)}T_1 = t - x_1) \\ & \quad \times \frac{x_1}{t} r \tilde{\Lambda}(dx_1) \mathbb{P}(\mathbb{B}^{(r+1)} \in dM_1, T(\mathbb{B}^{(r+1)}) \in d(t - x_1)) \\ &= \frac{x_2}{t - x_1} M_1(dx_2) \frac{x_1}{t} r \tilde{\Lambda}(dx_1) \mathbb{P}(\mathbb{B}^{(r+1)} \in dM_1, T(\mathbb{B}^{(r+1)}) \in d(t - x_1)). \end{aligned} \quad (3.26)$$

The second equality comes from (3.19) and the fact that \tilde{J}_2 is conditionally independent of \tilde{J}_1 given $\mathbb{B}_1^{(r)}$. In the last equality of (3.26), we used (3.21). Consequently, letting $\mathbb{M}_1 := \mathbb{M} - \delta_{x_1}$, we can compute

$$\begin{aligned} & \mathbb{P}(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, T(\mathbb{B}^{(r)}) \in dt) \\ &= \int_{M \in \mathbb{M}} \mathbb{P}(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, \mathbb{B}^{(r)} \in dM, T(\mathbb{B}^{(r)}) \in dt) \\ &= \int_{M_1 \in \mathbb{M}_1} \mathbb{P}(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, \mathbb{B}^{(r, x_1^-)} \in dM_1, T(\mathbb{B}^{(r, x_1^-)}) \in d(t - x_1)) \\ &= \frac{x_2}{t - x_1} \frac{x_1}{t} r \tilde{\Lambda}(dx_1) \int_{M_1 \in \mathbb{M}_1} M_1(dx_2) \mathbb{P}(\mathbb{B}^{(r+1)} \in dM_1, T(\mathbb{B}^{(r+1)}) \in d(t - x_1)) \\ &= \frac{x_2}{t - x_1} \frac{x_1}{t} r \tilde{\Lambda}(dx_1) \int_{M_1 \in \mathbb{M}_1} (r+1) \tilde{\Lambda}(dx_2) \mathbb{P}_{r+1}^{(x_2)}(dM_1, d(t - x_1)), \end{aligned} \quad (3.27)$$

where $\mathbb{P}_{r+1}^{(x_2)}(dM, dt) = \mathbb{P}(\xi \in dM, T(\xi) \in dt)$ with ξ is a random point process with law $\mathbb{P}_{r+1}^{(x_2)}$. The last equality in (3.27) comes from (3.22).

Let $\mathbb{M}_2 := \mathbb{M}_1 - \delta_{x_2}$, and change variable to $M_2 = M_1 - \delta_{x_2}$ as in (3.24), to get

the RHS of (3.27) equal to

$$\begin{aligned} & \frac{x_2}{t-x_1} \frac{x_1}{t} r \tilde{\Lambda}(\mathrm{d}x_1) (r+1) \tilde{\Lambda}(\mathrm{d}x_2) \\ & \quad \times \int_{M_2 \in \mathbb{M}_2} \mathrm{P}\{\mathbb{B}^{(r+2)} \in \mathrm{d}M_2, T(\mathbb{B}^{(r+2)}) \in \mathrm{d}(t-x_1-x_2)\} \\ & = r(r+1) \frac{x_2}{t-x_1} \frac{x_1}{t} \tilde{\Lambda}(\mathrm{d}x_1) \tilde{\Lambda}(\mathrm{d}x_2) \mathrm{P}\{^{(r+2)}T \in \mathrm{d}(t-x_1-x_2)\}. \end{aligned}$$

Here we note that $\mathrm{P}(\mathbb{B}^{(r+2)} \in \mathbb{M}_2) = 1$ as $\mathbb{B}^{(r+2)}$ has a diffuse mean measure, hence

$$\mathrm{P}(\mathbb{B}^{(r+2)}(\{x_1\}) > 0) = \mathrm{P}(\mathbb{B}^{(r+2)}(\{x_2\}) > 0) = 0.$$

By a similar argument, we can show that for each $n \in \mathbb{N}$, $x_i \in (0, 1)$, $t > \sum_{i=1}^n x_i$,

$$\begin{aligned} & \mathrm{P}(\tilde{J}_1 \in \mathrm{d}x_1, \dots, \tilde{J}_n \in \mathrm{d}x_n, T(\mathbb{B}^{(r)}) \in \mathrm{d}t) \\ & = \prod_{i=1}^n \frac{(r+i-1)x_i \tilde{\Lambda}(\mathrm{d}x_i)}{t - \sum_{j=1}^{i-1} x_j} \mathrm{P}\left(^{(r+n)}T \in \mathrm{d}\left(t - \sum_{i=1}^n x_i\right)\right), \end{aligned}$$

and this is the same as (3.25). \square

Next we make use of (3.25) to derive the joint densities of the size-biased quantities in (3.13)–(3.15). Write $\Theta(x) = \alpha x^{-\alpha} \mathbf{1}_{\{0 < x < 1\}}$.

Proposition 3.5. *Fix $r \in \mathbb{N}$.*

(i) *The joint density of $(^{(r)}T, ^{(r)}T_1, ^{(r)}T_2, \dots, ^{(r)}T_n)$ with respect to Lebesgue measure is, for $t_0 > t_1 > \dots > t_n > 0$ and $n \in \mathbb{N}$,*

$$f(t_0, t_1, \dots, t_n) = r^{(n)} g_{r+n}(t_n) \prod_{i=0}^{n-1} \frac{\Theta(t_i - t_{i+1})}{t_i}. \quad (3.28)$$

(ii) *The sequence $(^{(r)}T, ^{(r)}T_1, ^{(r)}T_2, \dots)$ is a (non-homogeneous) Markov Chain with transition density, for $t_n > t_{n+1} > 0$ and $n \in \mathbb{N}_0$ (recall $^{(r)}T_0 \equiv ^{(r)}T$):*

$$\mathrm{P}\left(^{(r)}T_{n+1} \in \mathrm{d}t_{n+1} \mid ^{(r)}T_n = t_n\right) = (r+n) \frac{\Theta(t_n - t_{n+1})}{t_n} \frac{g_{r+n+1}(t_{n+1})}{g_{r+n}(t_n)} \mathrm{d}t_{n+1}. \quad (3.29)$$

(iii) *The joint density of $(^{(r)}T_n, ^{(r)}U_1, ^{(r)}U_2, \dots, ^{(r)}U_n)$ is, for $t_n > 0$, $0 < u_i < 1$, $1 \leq i \leq n$, and $n \in \mathbb{N}$,*

$$h(t_n, u_1, \dots, u_n) = \frac{r^{(n)}}{K_n} g_{r+n}(t_n) t_n^{-n\alpha} \prod_{i=1}^n \frac{\Gamma(i\alpha + 1 - \alpha)}{\Gamma(i\alpha)\Gamma(1 - \alpha)} u_i^{i\alpha-1} \bar{u}_i^{-\alpha} \mathbf{1}_{\{t_n < \prod_{j=i}^n u_j / \bar{u}_i\}}, \quad (3.30)$$

where $\bar{u}_i = 1 - u_i$, and

$$K_n = \frac{\prod_{i=0}^{n-1} \Gamma(1 + i\alpha)}{\alpha^n \Gamma^n(1 - \alpha) \prod_{i=1}^n \Gamma(i\alpha)} = \frac{\Gamma(n+1)}{\Gamma^n(1 - \alpha) \Gamma(n\alpha + 1)}. \quad (3.31)$$

Remark 3.7. A routine calculation shows that

$$\frac{1}{\Gamma(n\alpha)} \int_0^\infty t^{n\alpha-1} \left(1 + \int_0^\infty (1 - e^{-tx}) \alpha x^{-\alpha-1} dx \right)^{-r-n} dt = \frac{K_n}{r(n)}. \quad (3.32)$$

Notice that the inner integration in (3.32) is over $x \in (0, \infty)$, whereas that in (3.17) is over $x \in (0, 1)$, and our $\Theta(x)$ is restricted to $(0, 1)$, whereas that of PPY (1992, Eq. (2.b)) is not. This is a reflection of the truncation induced by eliminating the large points. Still, the K_n in (3.31) and (3.32) exactly equals the K_n in Eq. (2.n) of PPY (1992), provided we set the stable scaling constant c in their Eq. (2.i) equal to $\Gamma(1 - \alpha)$. In both notations, $K_n = E(S_1^{-n\alpha})$ (see also Eq. (30) of PY (1997)).

In general, we have the following relation:

$$K_n = r^{(n)} \int_{u_1=0}^1 \cdots \int_{u_n=0}^1 \left\{ \int_{t_n=0}^{d(u_1, \dots, u_n)} t_n^{-n\alpha} g_{r+n}(t_n) dt_n \right\} \prod_{i=1}^n f_{B_{i\alpha, 1-\alpha}}(u_i) du_1 \cdots du_n, \quad (3.33)$$

where the function $d(u_1, \dots, u_n) = \min_{1 \leq i \leq n} \prod_{j=i}^n u_j / \bar{u}_i$ for $0 < u_i < 1$, $1 \leq i \leq n$, $n \in \mathbb{N}$, and $f_{B_{a,b}}$ is the density of a Beta(a,b) distribution as defined in (3.5).

Proof of Proposition 3.5: (i): By change of variable in (3.25), we have

$$\begin{aligned} & P\left({}^{(r)}T \in dt_0, {}^{(r)}T_1 \in dt_1, {}^{(r)}T_2 \in dt_2, \dots, {}^{(r)}T_n \in dt_n\right) \\ &= P\left({}^{(r)}T \in dt_0, \tilde{J}_1 \in d(t_0 - t_1), \tilde{J}_2 \in d(t_1 - t_2), \dots, \tilde{J}_n \in d(t_{n-1} - t_n)\right) \\ &= r^{(n)} \prod_{i=0}^{n-1} \frac{\Theta(t_i - t_{i+1})}{t_i} g_{r+n}(t_n) dt_0 dt_1 \cdots dt_n. \end{aligned}$$

This proves (3.28). Part (ii) follows immediately from Part (i):

$$\begin{aligned} & P\left({}^{(r)}T_{n+1} \in dt_{n+1} \mid {}^{(r)}T = t_0, {}^{(r)}T_1 = t_1, {}^{(r)}T_2 = t_2, \dots, {}^{(r)}T_n = t_n\right) \\ &= (r+n) \frac{\Theta(t_n - t_{n+1})}{t_n} \frac{g_{r+n+1}(t_{n+1})}{g_{r+n}(t_n)} dt_{n+1}, \end{aligned}$$

which does not depend on t_0, t_1, \dots, t_{n-1} . Thus (3.29) is established.

(iii) To show (3.30), we first consider the case $n = 2$. Note that

$$h(t_2, u_1, u_2) = f\left(\frac{t_2}{u_1 u_2}, \frac{t_2}{u_2}, t_2\right) t_2^2 u_1^{-2} u_2^{-3};$$

here $t_2^2 u_1^{-2} u_2^{-3}$ is the Jacobian from the change of variables. Expanding the expres-

sion in (3.28) with $\Theta(x) = \alpha x^{-\alpha} \mathbf{1}_{\{0 < x < 1\}}$, we get

$$\begin{aligned} h(t_2, u_1, u_2) &= r^{(2)} g_{r+2}(t_2) u_1^{-1} u_2^{-1} \Theta\left(\frac{t_2}{u_1 u_2} \bar{u}_1\right) \Theta\left(\frac{t_2}{u_2} \bar{u}_2\right) \\ &= r^{(2)} g_{r+2}(t_2) \alpha^2 t_2^{-2\alpha} (u_2^{2\alpha-1} \bar{u}_2^{-\alpha}) (u_1^{\alpha-1} \bar{u}_1^{-\alpha}) \mathbf{1}_{\{t_2 \bar{u}_2 / u_2 < 1\}} \mathbf{1}_{\{t_2 \bar{u}_1 / (u_1 u_2) < 1\}} \\ &= \frac{r^{(2)}}{K_2} g_{r+2}(t_2) t_2^{-2\alpha} \cdot \left[\frac{\Gamma(1+\alpha)}{\Gamma(2\alpha)\Gamma(1-\alpha)} u_2^{2\alpha-1} \bar{u}_2^{-\alpha} \right] \times \\ &\quad \times \left[\frac{\Gamma(1)}{\Gamma(\alpha)\Gamma(1-\alpha)} u_1^{\alpha-1} \bar{u}_1^{-\alpha} \right] \mathbf{1}_{\{t_2 < u_2 / \bar{u}_2\}} \mathbf{1}_{\{t_2 < u_1 u_2 / \bar{u}_1\}}, \end{aligned}$$

where

$$K_2 = \frac{\prod_{i=0}^1 \Gamma(1+i\alpha)}{\alpha^2 \Gamma^2(1-\alpha) \prod_{i=1}^2 \Gamma(i\alpha)}.$$

This formula can be generalised to $n \geq 2$ in a similar way, and (3.30) follows. \square

To complete this section our final theorem gives formulae for the distributions of the size-biased sequence constructed from the sequence in (3.3), and for the residual fractions in (3.14)–(3.15).

Theorem 3.1. (i) For each $r \in \mathbb{N}$ let $(V_n^{(r)})_{n \in \mathbb{N}}$ have a $\text{PD}_\alpha^{(r)}$ distribution as defined in (3.3), with corresponding size-biased sequence $(\tilde{V}_n^{(r)})$. Then for each $n \in \mathbb{N}$ the joint density of $(\tilde{V}_1^{(r)}, \dots, \tilde{V}_n^{(r)}, {}^{(r)}T)$ with respect to Lebesgue measure is

$$p_r(v_1, \dots, v_n, t) = r^{(n)} \alpha^n \times \prod_{i=1}^n \frac{v_i^{-\alpha}}{\bar{v}_{i-1}} \mathbf{1}_{\{v_i < 1/t\}} \times t^{-n\alpha} g_{r+n}(t \bar{v}_n), \quad (3.34)$$

where $t > 0$, $0 < v_i < 1$ are such that $\sum_{i=1}^n v_i < 1$, $\bar{v}_0 \equiv 1$, and, for each $i \geq 1$, $\bar{v}_i = 1 - v_1 - \dots - v_i$.

(ii) The joint distribution of ${}^{(r)}T_n$ and ${}^{(r)}U_1, {}^{(r)}U_2, \dots, {}^{(r)}U_n$ can be written as

$$\{{}^{(r)}T_n, {}^{(r)}U_1, {}^{(r)}U_2, \dots, {}^{(r)}U_n\} \stackrel{\text{D}}{=} \{Y_{d(U_1, \dots, U_n)}, U_1, U_2, \dots, U_n\}, \quad (3.35)$$

where the (U_i) are independent $\text{Beta}(\alpha, 1-\alpha)$ rvs, independent of $\mathbb{B}^{(r)}$, the function $d(u_1, \dots, u_n)$ is as defined in (3.33), and, for each $c > 0$, $Y_c \stackrel{\text{D}}{=} ({}^{(r+n)}T)^{-n\alpha} \mathbf{1}_{\{({}^{(r+n)}T) < c\}}$.

Proof of Theorem 3.1: (i) Identify the size-biased $\tilde{V}_i^{(r)}$ with the points $\tilde{J}_r(i)$ in (3.11) normalised by their sum $T(\mathbb{B}^{(r)})$. Then change variable in (3.25) to $v_i = x_i/t$ and substitute for $\tilde{\Lambda}$ to get

$$\begin{aligned} & \text{P}(\tilde{V}_1^{(r)} \in dv_1, \dots, \tilde{V}_n^{(r)} \in dv_n, T(\mathbb{B}^{(r)}) \in dt) \\ &= \text{P}(\tilde{J}_r(1) \in t dv_1, \dots, \tilde{J}_r(n) \in t dv_n, {}^{(r)}T \in dt) \\ &= r^{(n)} \alpha^n \prod_{i=1}^n \frac{t \cdot t^{-\alpha} v_i^{-\alpha} dv_i}{t - \sum_{j=1}^{i-1} t v_j} \times \text{P}\left({}^{(r+n)}T \in d\left(t - \sum_{i=1}^n x_i\right)\right) \quad (\text{by (3.25)}) \\ &= r^{(n)} \alpha^n t^{-n\alpha} \prod_{i=1}^n \frac{v_i^{-\alpha} dv_i}{\bar{v}_{i-1}} \text{P}({}^{(r+n)}T \in dt \bar{v}_n), \end{aligned}$$

Recall that $g_{r+n}(\cdot)$ is the density of $^{(r+n)}T$ (see (3.16)), to complete the proof of (3.34).

For Part (ii), the representation in (3.35) is immediate from Proposition 3.5 and (3.33), with the vector on the RHS of (3.35) having the structure specified. \square

Remark 3.8. (i) There are some quite involved manipulations in obtaining the above formulae. As a check on the calculations, in the Appendix we give direct verifications that (3.28) and (3.34) are probability densities (integrate to 1).

(ii) For a stick-breaking representation, we can solve (3.14) and (3.15) to get

$$\tilde{V}_n^{(r)} = (1 - {}^{(r)}U_n) \prod_{i=1}^{n-1} {}^{(r)}U_i. \quad (3.36)$$

The joint distribution of $({}^{(r)}U_i)_{1 \leq i \leq n}$ can be computed from (3.35), in which we note that U_1, U_2, \dots, U_n are individually independent but dependence overall is introduced via the connection with the Y term. In this respect the result is different from the PD_α situation, as we would expect, but the distribution of $\tilde{V}_n^{(r)}$ as given by (3.36) is sufficiently explicit to enable computations or simulations.

(iii) (3.34) generalises the corresponding version for $\text{PD}_\alpha = \text{PD}(\alpha, 0)$ in PY (1997, Prop. 47).

(iv) When we sample from a Poisson process, the various quantities in Proposition 3.5 are computed in PPY (1992, Theorem 2.1).

(v) Although motivated by the idea of trimming an integer number r of large jumps, our formulae once derived are valid for $r > 0$, and available for modelling purposes in this generality.

To conclude this section we expand briefly on the differences as well as the similarities between the PD_α and $\text{PD}_\alpha^{(r)}$ approaches. In both cases, start with a stable(α) subordinator S with ranked jumps $\Delta S_1^{(1)} \geq \Delta S_1^{(2)} \geq \dots$. The sequence $(\Delta S_1^{(i)} / S_1)_{i \geq 1}$ then has a PD_α distribution. We can think of these as the points from a Poisson point process with intensity measure $\Lambda(dx) = \alpha x^{-\alpha-1} dx$, normalised by their sum. For $\text{PD}_\alpha^{(r)}$, the analogous process is the negative binomial point process $\mathcal{BN}(r, \tilde{\Lambda})$ formed from ratios of jumps rather than from the jumps themselves, i.e.,

$$\mathbb{B}^{(r)} = \sum_{i \geq 1} \delta_{J_r(i)}, \quad \text{with} \quad J_r(i) = \frac{\Delta S_1^{(r+i)}}{\Delta S_1^{(r)}}, \quad i \in \mathbb{N}.$$

The normalised jumps on which a size-biased version is based are

$$\frac{J_r(i)}{\sum_{\ell \geq 1} J_r(\ell)} = \frac{\Delta S_1^{(r+i)}}{\Delta S_1^{(r)}} \bigg/ \frac{{}^{(r)}S_1}{\Delta S_1^{(r)}} = \frac{\Delta S_1^{(r+i)}}{{}^{(r)}S_1}, \quad i \in \mathbb{N}, \quad (3.37)$$

and the sequence formed from these has a $\text{PD}_\alpha^{(r)}$ distribution, as we define it, on the infinite simplex.

We may set $r = 0$ in (3.3) to have the distribution of $(V_n^{(r)})_{n \in \mathbb{N}}$, that is, $\text{PD}_\alpha^{(r)}$, reduce to that of $(V_n)_{n \in \mathbb{N}}$, that is, PD_α . But we cannot take $r = 0$ in (3.37) with the idea that the size-biased distribution associated with $\text{PD}_\alpha^{(r)}$ might then reduce to the one associated with PD_α . Our analysis proceeds via the process $\mathbb{B}^{(r)}$, which is not defined for $r = 0$ (its points $J_r(i)$ are not defined for $r = 0$). Setting $r = 0$ in formulae such as (3.28), (3.30), (3.34), etc., which result from an analysis of $\mathbb{B}^{(r)}$, is not permissible.

4 Poisson Point Processes and Ratios of Ordered Jumps

In this section we establish the framework of Poisson point processes needed to prove the Lévy process results in Section 2. We draw on methodology of BFM (2016). The corresponding $(\Delta_t)_{t \geq 0}$ in their paper is the point process of jumps of a Lévy process, but their setup is quite general.

Let Π be a Borel measure on $(0, \infty)$, locally finite at infinity, and let

$$\mathbb{D} = \sum_{s > 0} \delta_{(s, \Delta_s)}$$

be a Poisson point process on $[0, \infty) \times (0, \infty)$ with intensity measure $dt \times \Pi(dx)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The measure Π need not be a Lévy measure but as a special case $(\Delta_t)_{t \geq 0}$ could be the jump process of a Lévy process on \mathbb{R} , as we have it in Section 2. In any case Π has finite-valued tail function $\overline{\Pi} : (0, \infty) \rightarrow (0, \infty)$, defined by

$$\overline{\Pi}(x) := \Pi\{(x, \infty)\}, \quad x > 0,$$

a right-continuous, non-increasing function with $\overline{\Pi}(+\infty) = 0$. Assume throughout that $\Pi\{(0, \infty)\} = \overline{\Pi}(0+) = \infty$, so there are infinitely many non-zero points of $\Delta = (\Delta_t)_{t \geq 0}$ in any finite interval, a.s. Let

$$\overline{\Pi}^\leftarrow(x) = \inf\{y > 0 : \overline{\Pi}(y) \leq x\}, \quad x > 0,$$

be the right-continuous inverse of $\overline{\Pi}$.

BFM (2016) used a randomisation procedure to specify a sequence of points $\Delta_t^{(1)} \geq \Delta_t^{(2)} \geq \dots$ which represent the largest order statistics of \mathbb{D} , sampled on the time interval $[0, t]$, possibly with ties. The following distributional equivalence can be deduced from Lemma 1.1 of BFM (2016):

$$(\Delta_t^{(i)})_{1 \leq i \leq r} \stackrel{\text{D}}{=} (\overline{\Pi}^\leftarrow(\Gamma_i/t))_{1 \leq i \leq r}, \quad t > 0, \quad r \in \mathbb{N}, \quad (4.1)$$

where the Γ_i are Gamma($i, 1$) random variables.

We need some aspects of the convergence behaviour of ratios of the $\Delta_t^{(r)}$, as $t \downarrow 0$. The basic assumption is the regular variation of the tail function $\bar{\Pi}$. When $\bar{\Pi} \in RV_0(-\alpha)$ with $0 < \alpha < \infty$ or, equivalently, $\bar{\Pi}^\leftarrow \in RV_\infty(-1/\alpha)$, we have the easily verified convergence

$$t\bar{\Pi}(u\bar{\Pi}^\leftarrow(y/t)) \sim \frac{\bar{\Pi}(uy^{-1/\alpha}\bar{\Pi}^\leftarrow(1/t))}{\bar{\Pi}(\bar{\Pi}^\leftarrow(1/t))} \rightarrow u^{-\alpha}y \text{ as } t \downarrow 0, \text{ for all } u > 0. \quad (4.2)$$

We also use the notation

$$W_{r,n}(t) = \frac{\Delta_t^{(r+n)}}{\Delta_t^{(r)}}, \quad t > 0, \quad r, n \in \mathbb{N}. \quad (4.3)$$

Theorem 4.1. *Suppose $\bar{\Pi}(\cdot) \in RV_0(-\alpha)$ and take $x_k \geq 1$ for $0 \leq k \leq n-1$, $n = 2, 3, \dots$, $r \in \mathbb{N}$ and $z > 0$.*

(i) *Then, for $0 < u < 1$,*

$$\begin{aligned} & \lim_{t \downarrow 0} \mathbb{P} \left(\frac{\Delta_t^{(r+k)}}{\Delta_t^{(r+n)}} > x_k, \quad 0 \leq k \leq n-1 \mid W_{r,n}(t) = u, \Delta_t^{(r+n)} = \bar{\Pi}^\leftarrow(z/t) \right) \\ &= \lim_{t \downarrow 0} \mathbb{P} \left(\frac{\Delta_t^{(r+k)}}{\Delta_t^{(r+n)}} > x_k, \quad 0 \leq k \leq n-1 \mid W_{r,n}(t) = u \right) \\ &= \mathbf{1}_{\{u < x_0^{-1}\}} \mathbb{P}(J_{n-1}^{(k)}(u) > x_k, \quad 1 \leq k \leq n-1), \end{aligned} \quad (4.4)$$

where $J_{n-1}^{(1)}(u) \geq J_{n-1}^{(2)}(u) \dots \geq J_{n-1}^{(n-1)}(u)$ are distributed like the decreasing order statistics of $n-1$ independent and identically distributed random variables $(J_i(u))_{1 \leq i \leq n-1}$, each having the distribution in (2.5).

(ii) *For n, x_k, z as specified,*

$$\begin{aligned} & \lim_{t \downarrow 0} \mathbb{P} \left(\frac{\Delta_t^{(r+k)}}{\Delta_t^{(r+n)}} > x_k, \quad 0 \leq k \leq n-1 \mid \Delta_t^{(r+n)} = \bar{\Pi}^\leftarrow(z/t) \right) \\ &= \mathbb{P}(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) > x_k, \quad 1 \leq k \leq n-1, B_{r,n}^{1/\alpha} \leq x_0^{-1}), \end{aligned} \quad (4.5)$$

where the $J_i(u)$ are as in Part (i) and $B_{r,n}$ is a Beta(r, n) random variable independent of $(J_i(u))_{1 \leq i \leq n-1}$.

Remark 4.1. (i) The x_0 variable is superfluous in Part (i) of Theorem 4.1 but it is relevant in Part (ii).

(ii) If we make the convention that $B_{0,n} \equiv 0$ a.s., put $u = 0$ in (2.5), and identify $(J_i(0))$ with a sequence (L_i) of independent and identically distributed random variables each having the distribution tail

$$\mathbb{P}(L_1 > x) = \bar{\Lambda}(x) = x^{-\alpha}, \quad x > 1, \quad (4.6)$$

we get the case $r = 0$ of (4.5); namely, for n, x_k, z as specified,

$$\lim_{t \downarrow 0} \mathbb{P} \left(\frac{\Delta_t^{(k)}}{\Delta_t^{(n)}} > x_k, 1 \leq k \leq n-1 \middle| \Delta_t^{(n)} = \bar{\Pi}^{\leftarrow}(z/t) \right) = \mathbb{P}(L_{n-1}^{(k)} > x_k, 1 \leq k \leq n-1), \quad (4.7)$$

where $L_{n-1}^{(1)} \geq L_{n-1}^{(2)} \dots \geq L_{n-1}^{(n-1)}$ are the decreasing order statistics of $(L_i)_{1 \leq i \leq n-1}$. (4.7) can of course be proved directly.

(iii) The case $r \in \mathbb{N}$, $n = 1$, in Part (i) of Theorem 4.1, is covered by setting $n = r + 1$, $x_1 = \dots = x_{r-1} = 1$ when $r > 1$, in (4.7), to get

$$\lim_{t \downarrow 0} \mathbb{P} \left(\frac{\Delta_t^{(r)}}{\Delta_t^{(r+1)}} > x_r \middle| \Delta_t^{(r+1)} = \bar{\Pi}^{\leftarrow}(z/t) \right) = \mathbb{P}(L_r^{(r)} > x_r) = x_r^{-r\alpha}, \quad (4.8)$$

for $x_r \geq 1$ and $z > 0$. Here $L_r^{(r)} \stackrel{D}{=} \min_{1 \leq i \leq r} L_i$, where $(L_i)_{1 \leq i \leq r}$ are i.i.d. random variables, each having the distribution tail in (4.6). Note $L_r^{(r)} \stackrel{D}{=} 1/B_{r,1}^{1/\alpha}$.

(iv) Convergence of the conditional distributions in (4.5), (4.7), and (4.8), together with

$$\lim_{t \downarrow 0} \mathbb{P}(t\bar{\Pi}(\Delta_t^{(n+j)}) \leq z_{n+j}, 0 \leq j \leq r) = \mathbb{P}(\Gamma_{n+j} \leq z_{n+j}, 0 \leq j \leq r),$$

for $0 \leq z_{n+r} \leq \dots \leq z_n$, which follows easily from (4.1) and (4.2), implies convergence of the corresponding joint, and hence marginal, distributions. Since the right-hand sides of (4.5), (4.7), and (4.8) do not depend on z , independence obtains in the corresponding limiting joint distributions.

(v) Kevei and Mason (2014) consider limit distributions of ratios of consecutive jumps of subordinators. In their case the ratios are smaller than or equal to 1 whereas we need to consider ratios bigger than or equal to 1, and in a multivariate version, but the methods are much the same. Kevei and Mason prove converse results too, as also obtain in our case, as do boundary cases (slow and rapid variation), but we omit details here to save space.

Proof of Theorem 4.1: (i) Take $x_k \geq 1$ for $0 \leq k \leq n-1$, $n = 2, 3, \dots$, $r \in \mathbb{N}$, $z > 0$, and $u \in (0, 1)$, recall (4.3) and write the conditional probability

$$\begin{aligned} & \mathbb{P} \left(\frac{\Delta_t^{(r+k)}}{\Delta_t^{(r+n)}} > x_k, 0 \leq k \leq n-1 \middle| W_{r,n}(t) = u, \Delta_t^{(r+n)} = \bar{\Pi}^{\leftarrow}(z/t) \right) \\ &= \mathbf{1}_{\{u < x_0^{-1}\}} \\ & \times \mathbb{P} \left(\frac{\Delta_t^{(r+k)}}{\Delta_t^{(r+n)}} > x_k, 1 \leq k \leq n-1 \middle| \Delta_t^{(r)} = \bar{\Pi}^{\leftarrow}(z/t)/u, \Delta_t^{(r+n)} = \bar{\Pi}^{\leftarrow}(z/t) \right). \end{aligned} \quad (4.9)$$

Using properties of the 2-dimensional Poisson process, the probability in (4.9) is

$$P(J_{n-1}^{(k)}(t, u, z) > x_k \bar{\Pi}^{\leftarrow}(z/t), 1 \leq k \leq n-1),$$

where $J_{n-1}^{(1)}(t, u, z) \geq J_{n-1}^{(2)}(t, u, z) \geq \dots \geq J_{n-1}^{(n-1)}(t, u, z)$ are distributed like the decreasing order statistics of $n-1$ independent and identically distributed random variables $(J_i(t, u, z))_{1 \leq i \leq n-1}$, each having the distribution

$$P(J_1(t, u, z) \in dx) = \frac{\Pi(dx) \mathbf{1}\{\bar{\Pi}^{\leftarrow}(z/t) < x \leq \bar{\Pi}^{\leftarrow}(z/t)/u\}}{\bar{\Pi}(\bar{\Pi}^{\leftarrow}(z/t)) - \bar{\Pi}(\bar{\Pi}^{\leftarrow}(z/t)/u)}, \quad x > 0.$$

From (4.2) it follows that, as $t \downarrow 0$, for each $u \in (0, 1)$, $x > 1$ and $z > 0$,

$$\begin{aligned} & P(J_1(t, u, z) > x \bar{\Pi}^{\leftarrow}(z/t)) \\ &= \frac{t(\bar{\Pi}(x \bar{\Pi}^{\leftarrow}(z/t)) - \bar{\Pi}(\bar{\Pi}^{\leftarrow}(z/t)/u)) \mathbf{1}\{x \bar{\Pi}^{\leftarrow}(z/t) \leq \bar{\Pi}^{\leftarrow}(z/t)/u\}}{t\bar{\Pi}(\bar{\Pi}^{\leftarrow}(z/t)) - t\bar{\Pi}(\bar{\Pi}^{\leftarrow}(u/t)/u)} \\ &\rightarrow \frac{(x^{-\alpha}z - u^\alpha z) \mathbf{1}_{\{x^{-\alpha}z \geq u^\alpha z\}}}{z - u^\alpha z} = \frac{(x^{-\alpha} - u^\alpha) \mathbf{1}_{\{1 \leq x \leq 1/u\}}}{1 - u^\alpha} \\ &= P(J_1(u) > x), \end{aligned}$$

where $J_1(u)$ is a random variable with the distribution in (2.5). Returning to (4.9), we see that we have proved the equality of the first and third members of (4.4).

The RHS of (4.4) does not depend on z , consequently, multiplying each side of it by $d_z P(\Delta_t^{(r+n)} \leq \bar{\Pi}^{\leftarrow}(z/t))$ and integrating over $z \in (0, \infty)$, we obtain the equality of the second and third members of (4.4).

(ii) (4.5) follows by integrating each of the first and third members of (4.4) with respect to $P(W_{r,n}(t) \in du | \Delta_t^{(r+n)} = \bar{\Pi}^{\leftarrow}(z/t))$ for $0 < u < x_0^{-1}$ and using the next proposition. \square

Proposition 4.1. *Suppose $\bar{\Pi}(\cdot) \in RV_0(-\alpha)$ with $0 < \alpha < \infty$. Then $W_{r,n}(t) \xrightarrow{D} W_{r,n} \stackrel{D}{=} B_{r,n}^{1/\alpha}$, as $t \downarrow 0$.*

Proof of Proposition 4.1: We omit the details of this, which are quite similar to those of the previous proof. \square

5 Representations for Trimmed Lévy Processes

In the present section we revert to considering an arbitrary real valued Lévy process $(X_t)_{t \geq 0}$ set up as in Section 1 (see (1.1) and (1.2)). We consider only the “one-sided” trimming of BFM (2016), assuming throughout that $\bar{\Pi}^+(0+) = \infty$. Then there are

infinitely many positive jumps ΔX_t in any neighbourhood of 0, a.s. and we can define the ordered jumps $\Delta X_t^{(1)} \geq \Delta X_t^{(2)} \geq \dots$ at time $t > 0$ just as we did for the Δ_t in Section 4. The one-sided r -trimmed version of X is then defined as in (1.3), by subtracting $\Delta X_t^{(i)}$, $1 \leq i \leq r$, from X_t .

The starting point for this section is a general representation for the joint distribution of ${}^{(r)}X_t$ and its large jumps, given in BFM (2016), allowing for possible tied values in the jumps. Our first proposition derives some important properties from it.

Proposition 5.1. *Assume $\bar{\Pi}^+(0+) = \infty$. Take $r \in \mathbb{N}$, $v_1 \geq v_2 \geq \dots \geq v_r > 0$, $x \in \mathbb{R}$, $t > 0$. Then we have the Markov property*

$$\mathbb{P}({}^{(r)}X_t \leq x | \Delta X_t^{(i)} = \bar{\Pi}^{+, \leftarrow}(v_i), 1 \leq i \leq r) = \mathbb{P}({}^{(r)}X_t \leq x | \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v_r)), \quad (5.1)$$

which holds at points of increase $v_i > 0$, $1 \leq i \leq r$, of Π , and the identity

$$\mathbb{P}({}^{(r)}X_t \leq x | \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v)) = \mathbb{P}(X_t^v + G_t^v \leq x), \quad (5.2)$$

which holds at points of increase $v > 0$ of Π . In (5.2), $(X_t^v)_{t \geq 0}$ is a Lévy process, indexed by $v > 0$, having canonical triplet

$$(\gamma^v, \sigma^2, \Pi^v(dx)) := \left(\gamma - \mathbf{1}_{\{\bar{\Pi}^{+, \leftarrow}(v) \leq 1\}} \int_{\bar{\Pi}^{+, \leftarrow}(v) \leq x \leq 1} x \Pi(dx), \sigma^2, \Pi(dx) \mathbf{1}_{\{x < \bar{\Pi}^{+, \leftarrow}(v)\}} \right), \quad (5.3)$$

while $G_t^v := \bar{\Pi}^{+, \leftarrow}(v) Y_{t\kappa(v)}$ for $v > 0$, $t > 0$, with $\kappa(v) := \bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v)-) - v$ and $(Y_t)_{t \geq 0}$ a Poisson process with $EY_1 = 1$, independent of X .

Remark 5.1. The RHS of (5.2) does not depend on the index r , so it also equals $\mathbb{P}({}^{(1)}X_t \leq x | \Delta X_t^{(1)} = \bar{\Pi}^{+, \leftarrow}(v))$. Similarly in (5.8) and (5.9) below.

Proof of Proposition 5.1: To state the BFM (2016) representation, let $v > 0$ and introduce the Lévy process $(X_t^v)_{t \geq 0}$ having canonical triplet in (5.3), the Poisson process Y , and the quantities G_t^v and κ as in Proposition 5.1. Let $r \in \mathbb{N}$ and recall that (Γ_i) are Gamma(i) random variables, $i \in \mathbb{N}$. Assume that X , (Γ_i) and Y are independent as random elements. Then Lemma 1 p.2333 of BFM (2016) gives, for each $t > 0$,

$$({}^{(r)}X_t, \Delta X_t^{(1)}, \dots, \Delta X_t^{(r)}) \stackrel{D}{=} (X_t^{\Gamma_r/t} + G_t^{\Gamma_r/t}, \bar{\Pi}^{+, \leftarrow}(\Gamma_1/t), \dots, \bar{\Pi}^{+, \leftarrow}(\Gamma_r/t)).$$

From this we can compute, for $t > 0$, $u > 0$,

$$\mathbb{P}({}^{(r)}X_t \leq x, \Delta X_t^{(r)} \leq u) = \int_{\bar{\Pi}^{+, \leftarrow}(v) \leq u} \mathbb{P}(X_t^v + G_t^v \leq x) \mathbb{P}(\Gamma_r \in t dv)$$

$$= \int_{v \geq \bar{\Pi}^+(u)} P(X_t^v + G_t^v \leq x) P(\bar{\Pi}^+(\Delta X_t^{(r)}) \in dv). \quad (5.4)$$

The LHS of (5.4) equals $P({}^{(r)}X_t \leq x, \bar{\Pi}^+(\Delta X_t^{(r)}) \geq \bar{\Pi}^+(u))$, so we deduce

$$P({}^{(r)}X_t \leq x | \bar{\Pi}^+(\Delta X_t^{(r)}) = v) = P(X_t^v + G_t^v \leq x), \quad (5.5)$$

and the LHS of (5.5) equals $P({}^{(r)}X_t \leq x | \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v))$ at points of increase⁶ v of $\bar{\Pi}$. So we get (5.2). (5.1) follows similarly. \square

Using Proposition 5.1, the conditional characteristic functions of ${}^{(r)}X_t$ can be written as in the next corollary.

Corollary 5.1. *Assume $\bar{\Pi}^+(0+) = \infty$. For $v > 0$, $r \in \mathbb{N}$, $t > 0$, $\theta \in \mathbb{R}$,*

$$\begin{aligned} & E(e^{i\theta {}^{(r)}X_t} | \Delta X_t^{(1)}, \dots, \Delta X_t^{(r-1)}, \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v)) \text{ (when } r = 2, 3, \dots) \\ &= E(e^{i\theta {}^{(r)}X_t} | \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v)) \\ &= \exp \left(i\theta t \gamma^v - t\sigma^2 \theta^2 / 2 + t \int_{(-\infty, \bar{\Pi}^{+, \leftarrow}(v))} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \Pi(dx) \right) \\ &\quad \times \exp(t\kappa(v)(e^{i\theta \bar{\Pi}^{+, \leftarrow}(v)} - 1)), \end{aligned} \quad (5.6)$$

where γ^v is the shift constant defined in (5.3).

Suppose X is of bounded variation ($X \in bv$) with drift $d_X := \gamma - \int_{0 < |x| \leq 1} x \Pi(dx)$. Then the RHS of (5.6) can be replaced by (recall $\sigma^2 = 0$ when $X \in bv$)

$$\exp \left(i\theta t d_X + t \int_{(-\infty, \bar{\Pi}^{+, \leftarrow}(v))} (e^{i\theta x} - 1) \Pi(dx) \right) \times \exp(t\kappa(v)(e^{i\theta \bar{\Pi}^{+, \leftarrow}(v)} - 1)). \quad (5.7)$$

The next corollary follows immediately from Corollary 5.1. Recall the definitions of μ_X and μ_S in (2.4).

Corollary 5.2. *For $v > 0$, $\theta \in \mathbb{R}$, $t > 0$, $r \in \mathbb{N}$,*

$$\begin{aligned} & E \left(\exp \left(i\theta \frac{{}^{(r)}X_t - t\mu_X(\Delta X_t^{(r)})}{\Delta X_t^{(r)}} \right) \middle| \Delta X_t^{(1)}, \dots, \Delta X_t^{(r-1)}, \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v) \right) \\ &= E \left(\exp \left(i\theta \frac{{}^{(r)}X_t - t\mu_X(\Delta X_t^{(r)})}{\Delta X_t^{(r)}} \right) \middle| \Delta X_t^{(r)} = \bar{\Pi}^{+, \leftarrow}(v) \right) \\ &= \exp(-t\sigma^2 \theta^2 / 2 (\bar{\Pi}^{+, \leftarrow}(v))^2 + \\ &\quad + t \int_{(-\infty, 1)} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \Pi(\bar{\Pi}^{+, \leftarrow}(v) dx) \times \exp(t\kappa(v)(e^{i\theta} - 1)). \end{aligned} \quad (5.8)$$

⁶In general $\bar{\Pi}(x) = y$ does not imply $x = \bar{\Pi}^{\leftarrow}(y)$, but this does hold when x is a point of increase of $\bar{\Pi}$.

Suppose X is of bounded variation with drift d_X . Then (5.8) can be replaced by

$$\begin{aligned}
& E\left(\exp\left(i\theta\frac{^{(r)}X_t - td_X}{\Delta X_t^{(r)}}\right)\middle|\Delta X_t^{(1)}, \dots, \Delta X_t^{(r-1)}, \Delta X_t^{(r)} = \overline{\Pi}^{+, \leftarrow}(v)\right) \quad (r \geq 2) \\
&= E\left(\exp\left(i\theta\frac{^{(r)}X_t - td_X}{\Delta X_t^{(r)}}\right)\middle|\Delta X_t^{(r)} = \overline{\Pi}^{+, \leftarrow}(v)\right) \\
&= \exp\left(t \int_{(-\infty, 1)} (e^{i\theta x} - 1) \Pi(\overline{\Pi}^{+, \leftarrow}(v) dx)\right) \times \exp(t\kappa(v)(e^{i\theta} - 1)). \quad (5.9)
\end{aligned}$$

Proof of Corollaries 5.1 and 5.2: (5.6) follows from Proposition 5.1, using (5.3). Then (5.7) follows from (5.6) by rearranging the centering terms. (5.8) follows from (5.6) and (2.4), and (5.9) follows from (5.8). \square

Another formula follows from (5.1):

Corollary 5.3. Suppose X is of bounded variation with drift d_X . Then for $u \geq v > 0$, $\theta \in \mathbb{R}$, $t > 0$, $r \in \mathbb{N}$, $n \in \mathbb{N}$,

$$\begin{aligned}
& E\left(\exp\left(i\theta\frac{^{(r+n)}X_t - td_X}{\Delta X_t^{(r)}}\right)\middle|\Delta X_t^{(r)} = u, \Delta X_t^{(r+n)} = v\right) \\
&= \exp\left(t \int_{(-\infty, v)} (e^{i\theta x/u} - 1) \Pi(dx)\right) \times \exp(t\kappa(\overline{\Pi}^+(v))(e^{i\theta v/u} - 1)). \quad (5.10)
\end{aligned}$$

Remark 5.2. Many of the results in Sections 2 and 4 have antecedents in a discrete time environment, relating to extremes of i.i.d. sequences of random variables and their relation to random walks and trimmed sums. Out of a large literature we mention just: Darling (1956) (ratio of maximum to sum of stable rvs), Arov and Bobrov (1960) (role of the extreme terms in the sample sum), Smid and Stam (1975) (quotients of order statistics), Hall (1978) (extreme terms of a sample from a domain of attraction), Stadtmüller (1982), Stadtmüller and Trautner (1985) (Tauberian theorems, ratios of extremes), Griffin and Qazi (2002) (intermediate trimming of random walks), Teugels and Vanroelen (2002) (ratios and differences of order statistics), de Haan and Stadtmüller (2002), Silvestrov and Teugels (2004) (mixed max-sum processes).

6 Proofs for Section 2

Throughout this section X will be a Lévy process in the domain of attraction of a non-normal stable random variable. We only give proofs for $t \downarrow 0$. Thus the Lévy tail $\overline{\Pi}$ is regularly varying of index $-\alpha$, $\alpha \in (0, 2)$, at 0, and the balance condition (2.1) holds at 0. Since $a_+ > 0$ in (2.1), also $\overline{\Pi}^+ \in RV_0(-\alpha)$ at 0.

Proof of Theorem 2.1: First take $r \in \mathbb{N}$, $n = 2, 3, \dots$, and choose $x_1 \geq \dots \geq x_{n-1} \geq 1$, $x_n = 1$, $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$, and $v > 0$. For shorthand, write $M_t^{(r+n)}$ for $\mu_X(\Delta X_t^{(r+n)})$. We proceed by finding the limit as $t \downarrow 0$ of the conditional characteristic function

$$\begin{aligned} & \mathbb{E} \left(\exp \left(i \sum_{k=1}^n \frac{\theta_k {}^{(r)}X_t - t M_t^{(r+n)}}{\Delta X_t^{(r+k)}} \right) \middle| \Delta X_t^{(r+k)} = x_k \bar{\Pi}^{+, \leftarrow}(v/t), 1 \leq k \leq n \right) = \\ & \mathbb{E} \left(\exp \left(i \sum_{k=1}^n \frac{\theta_k {}^{(r)}X_t - t M_t^{(r+n)}}{\Delta X_t^{(r+n)}} \cdot \frac{x_n}{x_k} \right) \middle| \Delta X_t^{(r+k)} = x_k \bar{\Pi}^{+, \leftarrow}(v/t), 1 \leq k \leq n \right). \end{aligned} \quad (6.1)$$

Decompose ${}^{(r)}X_t$ as follows:

$$\frac{{}^{(r)}X_t}{\Delta X_t^{(r+n)}} = \sum_{k=1}^n \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} + \frac{{}^{(r+n)}X_t}{\Delta X_t^{(r+n)}}, \quad (6.2)$$

and recall the definitions of x_{n+} and $\tilde{\theta}_n$ in (2.8). Given the conditioning in (6.1), the first component on the RHS of (6.2) equals $\sum_{k=1}^n x_k/x_n = x_{n+}$. For the second component, apply Corollary 5.2 to replace the conditioning on $\Delta X_t^{(r+k)}$, $1 \leq k \leq n$, by conditioning on $\Delta X_t^{(r+n)}$. Then the RHS of (6.1) can be written as

$$e^{i\tilde{\theta}_n x_{n+}} \mathbb{E} \left(\exp \left(i\tilde{\theta}_n \frac{{}^{(r+n)}X_t - t M_t^{(r+n)}}{\Delta X_t^{(r+n)}} \right) \middle| \Delta X_t^{(r+n)} = x_n \bar{\Pi}^{+, \leftarrow}(v/t) = \bar{\Pi}^{+, \leftarrow}(v/t) \right). \quad (6.3)$$

Then by (5.8) with r replaced by $r+n$, θ replaced by $\tilde{\theta}_n$, v replaced by v/t , and $\sigma^2 = 0$, (6.3) equals

$$\begin{aligned} & e^{i\tilde{\theta}_n x_{n+}} \times \exp \left(\int_{(-\infty, 1)} (e^{i\tilde{\theta}_n x} - 1 - i\tilde{\theta}_n x \mathbf{1}_{\{|x| \leq 1\}}) t \Pi(\bar{\Pi}^{+, \leftarrow}(v/t) dx) \right) \\ & \times \exp(t\kappa(v/t)(e^{i\tilde{\theta}_n} - 1)). \end{aligned} \quad (6.4)$$

The term containing κ here (see Proposition 5.1) is negligible, as follows: $\bar{\Pi}^+$ in $RV_0(-\alpha)$ implies $\Delta \bar{\Pi}^+(x) := \bar{\Pi}^+(x-) - \bar{\Pi}^+(x) = o(\bar{\Pi}^+(x))$ as $x \downarrow 0$. Hence

$$\begin{aligned} t\kappa(v/t) &= t\bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v/t)-) - v \\ &\leq t \left(\bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v/t)-) - \bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v/t)) \right) \\ &= t\Delta \bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v/t)) \\ &= o \left(t\bar{\Pi}^+(\bar{\Pi}^{+, \leftarrow}(v/t)) \right) = o(1), \text{ as } t \downarrow 0. \end{aligned} \quad (6.5)$$

The limit of the second factor in (6.4) can be found straightforwardly using integration by parts and applying (4.2) and (2.1). So the expression in (6.4) tends as $t \downarrow 0$ to

$$e^{i\tilde{\theta}_n x_{n+}} \times \exp \left(v \int_{(-\infty, 1)} (e^{i\tilde{\theta}_n x} - 1 - i\tilde{\theta}_n x \mathbf{1}_{\{|x| \leq 1\}}) \Lambda(dx) \right). \quad (6.6)$$

Thus, by (6.1), to find the limit as $t \downarrow 0$ of

$$\mathbb{E} \exp \left(i \sum_{k=1}^n \frac{\theta_k^{(r)} X_t - t M_t^{(r+n)}}{\Delta X_t^{(r+k)}} \right),$$

we have to multiply (6.6) by the limit of

$$d_v \mathbb{P} \left(\frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} \in dx_k, 1 \leq k \leq n-1, \Delta X_t^{(r+n)} \leq \bar{\Pi}^{\leftarrow}(v/t) \right),$$

and integrate over v and the x_k . From (4.5) with $x_0 = 1$ we see that this limit does not depend on v , and putting the RHS of (4.5) together⁷ with the expression in (6.6) we can write the limiting characteristic function of the n -vector on the LHS of (2.9) as

$$\begin{aligned} & \int_{\mathbf{x}^\uparrow \geq 1} e^{i\tilde{\theta}_n x_n} \int_{v>0} \exp \left(v \int_{(-\infty, 1)} (e^{i\tilde{\theta}_n x} - 1 - i\tilde{\theta}_n x \mathbf{1}_{\{|x| \leq 1\}}) \Lambda(dx) \right) \mathbb{P}(\Gamma_{r+n} \in dv) \\ & \times \mathbb{P}(J_{n-1}^{(k)}(B_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1) \end{aligned} \quad (6.7)$$

(recall that $\int_{\mathbf{x}^\uparrow \geq 1}$ denotes integration over the region $\{x_1 \geq x_2 \geq \dots \geq x_{n-1} \geq 1\}$).

Note that, with Λ defined as in (2.2), $\bar{\Lambda}(x) \in RV_0(-\alpha)$ and $\bar{\Lambda}^+$ and $\bar{\Lambda}^-$ satisfy (2.1). So exactly the same calculation with $\Delta S_1^{(k)}$ replacing $\Delta X_t^{(k)}$, for $r \leq k \leq n$, and Λ replacing Π , shows that the characteristic function of the vector of stable ratios on the RHS of (2.9) equals (6.7).⁸

To derive (2.10), observe that the exponent inside the integral in (6.7) is the characteristic function of a Lévy process $(W_v)_{v \geq 0}$ having Lévy triplet $(0, 0, \Lambda(dx) \mathbf{1}_{(-\infty, 1)})$, that is, of a $\text{Stable}(\alpha)$ process with jumps truncated below 1. So the integral with respect to v in (6.7) is

$$\int_{v>0} \mathbb{E}(e^{i\tilde{\theta}_n W_v}) \mathbb{P}(\Gamma_{r+n} \in dv) = \mathbb{E}(e^{i\tilde{\theta}_n W_{\Gamma_{r+n}}}),$$

and thus we obtain (2.10).

The alternative representation in (2.11) is obtained by evaluating the dv integral in (6.7), resulting in (recall $\psi(\cdot)$ defined in (2.6)):

$$\mathbb{E} \exp \left(i \sum_{k=1}^n \frac{\theta_k^{(r)} X_t - t \mu_X(\Delta X_t^{(r+1)})}{\Delta X_t^{(r+k)}} \right)$$

⁷Use the result: $\int f_t(\omega) P_t(d\omega) \rightarrow \int f(\omega) P(d\omega)$ when $P_t \xrightarrow{w} P$ are probability measures and $f_t \rightarrow f$, f cts, $|f| \leq 1$. In (6.7), the f_t are characteristic functions and the limit distribution P in (6.7) is continuous in all its variables.

⁸This easy correspondence is the reason for adopting the nonstandard centering in (2.3).

$$\begin{aligned}
& \rightarrow \mathbb{E} \exp \left(i \sum_{k=1}^n \frac{\theta_k \left({}^{(r)}S_1 - \mu_S(\Delta S_1^{(r+1)}) \right)}{\Delta S_1^{(r+k)}} \right) \\
& = \int_{\mathbf{x}^\dagger \geq 1} e^{i\tilde{\theta}_n x_{n+}} \int_{v>0} \frac{v^{r+n-1} e^{-v(1-\psi(\tilde{\theta}_n))}}{\Gamma(r+n)} dv \mathbb{P}(J_{n-1}^{(k)}(\mathbf{B}_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1),
\end{aligned} \tag{6.8}$$

equal to the expression in (2.11).

In the case $r \in \mathbb{N}_0$, $n = 1$, similar working shows that (6.7) can be replaced by

$$\begin{aligned}
& \lim_{t \downarrow 0} \mathbb{E} \left(e^{i\theta \left({}^{(r)}X_t - t\mu_X(\Delta X_t^{(r+1)}) \right) / \Delta X_t^{(r+1)}} \right) \\
& = e^{i\theta} \int_{v>0} \exp \left(v \int_{(0,1)} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \Lambda(dx) \right) \mathbb{P}(\Gamma_{r+1} \in dv) \\
& = e^{i\theta} \mathbb{E}(e^{i\theta W_{\Gamma_{r+1}}}), \theta \in \mathbb{R}.
\end{aligned} \tag{6.9}$$

Finally, to prove (2.13), set $\theta_1 = \dots = \theta_{n-1} = 0$, $\theta_n = \theta$ (so $\tilde{\theta}_n = \theta$ and, recall, $x_{n+} = x_1 + \dots + x_{n-1} + 1$) in (6.8) to get the LHS of (2.13) equal to

$$\begin{aligned}
& \int_{\mathbf{x}^\dagger \geq 1} \frac{e^{i\theta x_{n+}}}{(1 - \psi(\theta))^{r+n}} \mathbb{P}(J_{n-1}^{(k)}(\mathbf{B}_{r,n}^{1/\alpha}) \in dx_k, 1 \leq k \leq n-1) \\
& = \frac{e^{i\theta}}{(1 - \psi(\theta))^{r+n}} \int_{0 < u < 1} \mathbb{E} \exp \left(i\theta \sum_{k=1}^{n-1} J_{n-1}^{(k)}(u) \right) \mathbb{P}(\mathbf{B}_{r,n}^{1/\alpha} \in du) \\
& = \frac{e^{i\theta}}{(1 - \psi(\theta))^{r+n}} \int_{0 < u < 1} (\mathbb{E} e^{i\theta J_1(u)})^{n-1} \mathbb{P}(\mathbf{B}_{r,n}^{1/\alpha} \in du) \\
& = \frac{e^{i\theta}}{(1 - \psi(\theta))^{r+n}} \mathbb{E}(\phi^{n-1}(\theta, \mathbf{B}_{r,n}^{1/\alpha})),
\end{aligned}$$

where $\phi(\theta, u) = \mathbb{E} e^{i\theta J_1(u)}$ as in (2.7) and we keep $|\theta| \leq \theta_0$, so $|\psi(\theta)| < 1$. Similarly, (6.9) can alternatively be written as $e^{i\theta}$ times the expression in (2.15). \square

Proof of Theorem 2.2: From (5.10) we obtain the Laplace transform

$$\begin{aligned}
& \mathbb{E} \exp \left(-\lambda \frac{{}^{(r+n)}X_t}{\Delta X_t^{(r)}} \right) = \\
& \int_{y>0} \int_{w>y} e^{-t \int_{(0,a)} (1 - e^{-\lambda x/b}) \Pi(dx) - t\kappa(w/t)(1 - e^{-\lambda a/b})} \mathbb{P}(\Gamma_r \in dy, \Gamma_{r+n} \in dw),
\end{aligned} \tag{6.10}$$

where $\lambda > 0$ and for brevity

$$a = a(w, t) := \overline{\Pi}^{\leftarrow}(w/t) \leq b = b(y, t) := \overline{\Pi}^{\leftarrow}(y/t), \quad t > 0, \quad w > y > 0$$

(we can write $\overline{\Pi}$ and $\overline{\Pi}^{\leftarrow}$ for $\overline{\Pi}^+$ and $\overline{\Pi}^{+, \leftarrow}$ in (5.10)). We derive an upper bound for the exponent in (6.10) as follows. Keep $0 < t \leq t_0$ for a fixed $t_0 > 0$, throughout.

First, the integral in the exponent of (6.10) is

$$\begin{aligned} t \int_{(0,a)} (1 - e^{-\lambda x/b}) \Pi(dx) &\leq t(\lambda/b) \int_0^a e^{-\lambda x/b} \bar{\Pi}(x) dx \quad (\text{integrate by parts}) \\ &= t\lambda \int_0^{a/b} e^{-\lambda x} \bar{\Pi}(bx) dx. \end{aligned} \quad (6.11)$$

Now $a(w, t) \rightarrow \bar{\Pi}^{\leftarrow}(+\infty) = 0$ as $w \rightarrow \infty$ or $t \downarrow 0$, and $b(y, t) \rightarrow \bar{\Pi}^{\leftarrow}(+\infty) = 0$ as $y \rightarrow \infty$ or $t \downarrow 0$. To compare the magnitudes of a and b we use the Potter bounds (Bingham, Goldie and Teugels (1987, p.25)). Since $\bar{\Pi} \in RV_0(-\alpha)$ with $0 < \alpha < 1$, given an $\eta \in (0, \min(\alpha, 1 - \alpha))$, there are constants $c > 0$ and $z_0 = z_0(\eta) > 0$ such that

$$\frac{\bar{\Pi}(\mu z)}{\bar{\Pi}(z)} \leq c\mu^{-\alpha-\eta} \text{ for all } \mu \in (0, 1), z \in (0, z_0); \quad (6.12)$$

and since $\bar{\Pi}^{\leftarrow} \in RV_{\infty}(-1/\alpha)$ we also have

$$\frac{\bar{\Pi}^{\leftarrow}(\mu z)}{\bar{\Pi}^{\leftarrow}(z)} \leq c\mu^{-1/\alpha+\eta} \text{ for all } \mu > 1, z > 1/z_0 \quad (6.13)$$

(where c and z_0 may be chosen the same in both cases and $\eta < \alpha < 1/\alpha$). Thus for $0 < x \leq a/b \leq 1$ and $0 < b \leq z_0$, using (6.12),

$$t\bar{\Pi}(bx) \leq ctx^{-\alpha-\eta}\bar{\Pi}(b) = ctx^{-\alpha-\eta}\bar{\Pi}(\bar{\Pi}^{\leftarrow}(y/t)) \leq cyx^{-\alpha-\eta},$$

and we have $b \leq z_0$ if $\bar{\Pi}^{\leftarrow}(y/t) \leq z_0$, i.e., if $y/t \geq \bar{\Pi}(z_0)$. For $w > y$ and $y/t \geq 1/z_0$, using (6.13),

$$\frac{a}{b} = \frac{\bar{\Pi}^{\leftarrow}(w/t)}{\bar{\Pi}^{\leftarrow}(y/t)} = \frac{\bar{\Pi}^{\leftarrow}(\frac{w}{y} \frac{y}{t})}{\bar{\Pi}^{\leftarrow}(\frac{y}{t})} \leq c \left(\frac{w}{y} \right)^{-1/\alpha+\eta} = c \left(\frac{y}{w} \right)^{1/\alpha-\eta}. \quad (6.14)$$

Keeping $y/t \geq z_1 := \bar{\Pi}(z_0) \vee (1/z_0)$, we now have by (6.11)

$$\begin{aligned} t \int_{(0,a)} (1 - e^{-\lambda x/b}) \Pi(dx) &\leq t\lambda \int_0^{a/b} e^{-\lambda x} \bar{\Pi}(bx) dx \\ &\leq c\lambda y \int_0^{a/b} x^{-\alpha-\eta} dx = \frac{c\lambda y}{1 - \alpha - \eta} \left(\frac{a}{b} \right)^{1-\alpha-\eta} \\ &\leq c'\lambda y \left(\frac{y}{w} \right)^{\beta} =: \lambda g_1(w, y), \end{aligned} \quad (6.15)$$

where $c' := c^{2-\alpha-\eta}/(1 - \alpha - \eta) > 0$ and $\beta := (1 - \alpha - \eta)(1/\alpha - \eta) > 0$.

Alternatively, when $y/t < z_1$, we have $b = \bar{\Pi}^{\leftarrow}(y/t) \geq \bar{\Pi}^{\leftarrow}(z_1)$, while $t \leq t_0$ implies $a = \bar{\Pi}^{\leftarrow}(w/t) \leq \bar{\Pi}^{\leftarrow}(w/t_0)$. Then

$$t \int_{(0,a)} (1 - e^{-\lambda x/b}) \Pi(dx) \leq t(\lambda/b) \int_{(0,a)} x \Pi(dx)$$

$$\begin{aligned}
&\leq t_0(\lambda/\bar{\Pi}^{\leftarrow}(z_1)) \int_{(0, \bar{\Pi}^{\leftarrow}(w/t_0))} x \Pi(dx) \\
&=: \lambda g_2(w).
\end{aligned} \tag{6.16}$$

Next, for the term containing κ in (6.10), note that we can have, for all $x > 0$,

$$\Delta \bar{\Pi}(x) = \bar{\Pi}(x-) - \bar{\Pi}(x) \leq \bar{\Pi}(x/2) - \bar{\Pi}(x) \leq c \bar{\Pi}(x),$$

as a consequence of the regular variation of $\bar{\Pi}$ at 0 (we can choose $c > 0$ the same as in (6.13)). Thus for all $t > 0$ and $w > 0$, using (6.5),

$$t\kappa(w/t) \leq t\Delta \bar{\Pi}(\bar{\Pi}^{\leftarrow}(w/t)) \leq ct\bar{\Pi}(\bar{\Pi}^{\leftarrow}(w/t)) \leq cw.$$

Then $t\kappa(w/t)(1 - e^{-\lambda a/b}) \leq cw\lambda a/b$. When $w > y$ and $y/t \geq 1/z_0$ this is no larger than $c^2\lambda w(y/w)^{1/\alpha-\eta}$ by (6.14). When $y/t < z_1$, so $b \geq \bar{\Pi}^{\leftarrow}(z_1)$, it is no larger than $cw\lambda \bar{\Pi}^{\leftarrow}(w/t_0)/\bar{\Pi}^{\leftarrow}(z_1)$. So an overall upper bound for the term containing κ in (6.10) is

$$t\kappa(w/t)(1 - e^{-\lambda a/b}) \leq \lambda g_3(w, y) := \max(c^2\lambda w(y/w)^{1/\alpha-\eta}, cw\lambda \bar{\Pi}^{\leftarrow}(w/t_0)/\bar{\Pi}^{\leftarrow}(z_1)). \tag{6.17}$$

Combine (6.15)–(6.17) to get an upper bound for the negative of the exponent in (6.10) of the form

$$\lambda g(w, y) := \lambda \max(g_1(w, y) + g_2(w), g_3(w, y)).$$

So, for all $0 < t \leq t_0$, $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E} \exp \left(-\lambda \frac{(r+n)X_t}{\Delta X_t^{(r)}} \right) &\geq \int_{y>0} \int_{w>y} e^{-t_0 \lambda g(w, y)} \mathbb{P}(\Gamma_r \in dy, \Gamma_{r+n} \in dw) \\
&= \mathbb{E} \left(e^{-t_0 \lambda g(\Gamma_{r+n}, \Gamma_r)} \right).
\end{aligned} \tag{6.18}$$

Now when $w \rightarrow \infty$, $g_1(w, y) \rightarrow 0$ for each $y > 0$ (see (6.15)) and $g_2(w) \rightarrow 0$ as $w \rightarrow \infty$ for each $y > 0$ because $\bar{\Pi}^{\leftarrow}(w) \rightarrow 0$ as $w \rightarrow \infty$ (see (6.16)); while $g_3(w, y) \rightarrow 0$ for each $y > 0$ because $\bar{\Pi}^{\leftarrow} \in RV_{\infty}(-1/\alpha)$ and $0 < \alpha < 1$ (see (6.17)).

Finally, since $\Gamma_{r+n} \xrightarrow{P} \infty$ as $n \rightarrow \infty$ for each $r \in \mathbb{N}$, we can let $n \rightarrow \infty$ and use Fatou's lemma in (6.18) to see that

$$\mathbb{E} \exp \left(-\lambda \frac{(r+n)X_t}{\Delta X_t^{(r)}} \right) \rightarrow 1, \text{ as } n \rightarrow \infty,$$

for each $r \in \mathbb{N}$, uniformly in $\lambda > 0$ and $t \in (0, t_0]$. We deduce convergence in probability in (2.19) uniformly in $t \in (0, t_0]$ from this, then since the LHS of (2.19) is monotone in n , we get the a.s. convergence. \square

Acknowledgements. We are grateful for some very helpful feedback from Peter Kevei and David Mason.

7 Appendix: (3.28) and (3.34) integrate to 1

As a check on the calculations, we give here a direct verification that (3.28) and (3.34) integrate to 1 in the case $n = 1$. An extension to larger n is straightforward.

Eq. (3.28) gives, for $n = 1$, $r \in \mathbb{N}$, $t_0 > t_1 > 0$,

$$f(t_0, t_1) = r g_{r+1}(t_1) \frac{\Theta(t_0 - t_1)}{t_0} = r \alpha g_{r+1}(t_1) \frac{(t_0 - t_1)^{-\alpha} \mathbf{1}_{\{t_0 - t_1 < 1\}}}{t_0}$$

Notice that

$$\int_{t_0=t_1}^{1+t_1} \frac{(t_0 - t_1)^{-\alpha}}{t_0} dt_0 = \int_0^1 \frac{t^{-\alpha}}{t + t_1} dt$$

so

$$\begin{aligned} \int_{t_1=0}^{\infty} \int_{t_0=t_1}^{1+t_1} f(t_0, t_1) dt_1 dt_0 &= r \alpha \int_{t_1=0}^{\infty} g_{r+1}(t_1) \int_{t=0}^1 \frac{t^{-\alpha}}{t + t_1} dt \\ &= r \alpha \int_{t=0}^1 t^{-\alpha} dt \int_{t_1=0}^{\infty} \frac{g_{r+1}(t_1)}{t + t_1} dt_1. \end{aligned} \quad (7.1)$$

Introduce an integral over λ and then substitute from (3.16) to write the last expression as

$$\begin{aligned} &r \alpha \int_{t=0}^1 t^{-\alpha} dt \int_{t_1=0}^{\infty} \int_{\lambda=0}^{\infty} e^{-\lambda(t+t_1)} d\lambda g_{r+1}(t_1) dt_1 \\ &= r \alpha \int_{t=0}^1 t^{-\alpha} dt \int_{\lambda=0}^{\infty} e^{-\lambda t} \int_{t_1=0}^{\infty} e^{-\lambda t_1} g_{r+1}(t_1) dt_1 d\lambda \\ &= r \alpha \int_{t=0}^1 t^{-\alpha} dt \int_{\lambda=0}^{\infty} \frac{e^{-\lambda t} d\lambda}{\left(1 + \int_0^1 (1 - e^{-\lambda x}) \tilde{\Lambda}(dx)\right)^{r+1}}. \end{aligned}$$

The last equation used (3.17). The final integral can be evaluated as

$$\begin{aligned} r \int_{\lambda=0}^{\infty} \frac{\alpha \int_{t=0}^1 t^{-\alpha} e^{-\lambda t} dt d\lambda}{\left(1 + \int_0^1 (1 - e^{-\lambda x}) \tilde{\Lambda}(dx)\right)^{r+1}} &= r \frac{\left(1 + \int_0^1 (1 - e^{-\lambda x}) \tilde{\Lambda}(dx)\right)^{-r}}{-r} \Big|_0^{\infty} \\ &= 1. \end{aligned}$$

Next we give a direct verification that (3.34) integrates to 1 in the case $n = 1$.

$$\begin{aligned} &r \alpha \int_{v_1=0}^1 v_1^{-\alpha} \int_{t=0}^{1/v_1} t^{-\alpha} g_{r+1}(t(1 - v_1)) dt \\ &= r \alpha \int_{v_1=0}^1 v_1^{-\alpha} (1 - v_1)^{\alpha-1} \int_{t=0}^{(1-v_1)/v_1} t^{-\alpha} g_{r+1}(t) dt \\ &= r \alpha \int_{y=0}^{\infty} \frac{y^{\alpha-1}}{1+y} dy \int_{t=0}^y t^{-\alpha} g_{r+1}(t) dt \quad (y = (1 - v_1)/v_1) \\ &= r \alpha \int_{t=0}^{\infty} t^{-\alpha} g_{r+1}(t) dt \int_{y=t}^{\infty} \frac{y^{\alpha-1}}{1+y} dy \end{aligned}$$

$$\begin{aligned}
&= r\alpha \int_{t=0}^{\infty} g_{r+1}(t)dt \int_{y=1}^{\infty} \frac{y^{\alpha-1}}{1+ty} dy \quad (y = y/t) \\
&= r\alpha \int_{y=1}^{\infty} y^{\alpha-1} dy \int_{t=0}^{\infty} \frac{1}{1+ty} g_{r+1}(t) dt \\
&= r\alpha \int_{x=0}^1 x^{-\alpha} dx \int_{t=0}^{\infty} \frac{g_{r+1}(t)}{x+t} dt \quad (x = 1/y).
\end{aligned}$$

The last is the RHS of (7.1) which equals 1.

References

- Arov, D. & Bobrov, A. (1960). The extreme terms of a sample and their role in the sum of independent variables. *Theory Probab. Appl.*, 5(4), 377–396.
- Bertoin, J. (2006). *Random Fragmentation and Coagulation Processes*, volume 102 of *Cambridge studies in advanced mathematics*. Cambridge University Press, Cambridge.
- Bingham, N. H., Goldie, C. M., & Teugels, J. L. (1987). *Regular Variation*. Cambridge University Press.
- Buchmann, B., Fan, Y., & Maller, R. A. (2016). Distributional representations and dominance of a Lévy process over its maximal jump processes. *Bernoulli*, 22(4), 2325–2371.
- Daley, D. J. & Vere-Jones, D. (1988). *An Introduction to the Theory of Point Processes*. Springer-Verlag, New York.
- Darling, D. A. (1956). The maximum of sums of stable random variables. *Trans. Amer. Math. Soc.*, 83(1), 164–169.
- de Haan, L. & Stadtmüller, U. (2002). Dominated variation and related concepts and Tauberian theorems for Laplace transforms. *Journal of Mathematical Analysis and Applications*, 108, 344–365.
- Feng, S. (2010). *The Poisson-Dirichlet Distribution and Related Topics: Models and Asymptotic Behaviours*. Probability and its Applications. Springer.
- Fitzsimmons, P., Pitman, J., & Yor, M. (1993). Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992*, Progress in Probability (pp. 101–134). Birkhäuser Boston.
- Gregoire, G. (1984). Negative binomial distributions for point processes. *Stochastic Process. Appl.*, 16(2), 179–188.
- Griffin, P. S. & Qazi, F. S. (2002). Limit laws of modulus trimmed sums. *Ann. Probab.*, 30(3), 1466–1485.
- Hall, P. (1978). On the extreme terms of a sample from the domain of attraction of a stable law. *J. London Math. Soc.*, 18(2), 181–191.

- James, L. F. (2013). Stick-breaking PG(α, ζ)-generalized gamma processes. *arXiv:1308.6570[math.PR]*.
- James, L. F. (2015). Generalized Mittag Leffler distributions arising as limits in preferential attachment models. *arXiv:1509.07150[math.PR]*.
- Kevei, P. & Mason, D. M. (2014). The limit distribution of ratios of jumps and sums of jumps of subordinators. *Lat. Am. J. Probab. Math. Stat.*, 11(2), 631–642.
- Kingman, J. F. C. (1975). Random discrete distributions. *J. R. Stat. Soc. Series B Stat. Methodol.*, 37(1), 1–22.
- Perman, M. (1990). *Random discrete distributions derived from subordinators*. PhD Thesis. Dept. of Statist., Univ. of California (Berkeley, CA).
- Perman, M. (1993). Order statistics for jumps of normalised subordinators. *Stochastic Process. Appl.*, 46(2), 267–281.
- Perman, M., Pitman, J., & Yor, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields*, 92(1), 21–39.
- Pitman, J. (1995). Random discrete distributions invariant under size-biased permutations. *Adv. App; Prob.*, 28, 525–539.
- Pitman, J. (2003). Poisson-Kingman partitions. In *Science and Statistics: A Festschrift for Terry Speed*, volume 40 of *Lect. Notes Monogr. Ser.* (pp. 1–34). Institute of Mathematical Statistics.
- Pitman, J. & Yor, M. (1992). Arcsine laws and interval partitions derived from a stable subordinator. *Proc. Lond. Math. Soc.*, 3(2), 326–356.
- Pitman, J. & Yor, M. (1997). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Annals of Probability*, 25(2), 855–900.
- Resnick, S. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer-Verlag.
- Sato, K. I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.
- Silvestrov, D. S. & Teugels, J. L. (2004). Limit theorems for mixed max-sum processes with renewal stopping. *Annals of Applied Probability*, 14(4), 1838–1868.
- Smid, B. & Stam, A. (1975). Convergence in distribution of quotients of order statistics. *Stochastic Processes and their Applications*, 3(3), 287–292.
- Stadt Müller, U. (1982). Tauberian theorems for Laplace and Stieltjes transforms. *Journal of Mathematical Analysis and Applications*, 86, 146–156.

- Stadtmüller, U. & Trautner, R. (1985). Ratio Tauberian theorems for Laplace transforms without monotonicity assumptions. *Quart. J. Math. Oxford*, 36, 363–381.
- Teugels, J. L. & Vanroelen, G. (2002). Convergence of ratios and differences of two order statistics. *Publications de l'Institut Mathématique*, 71(85), 113–122.